WS08 [4.2.2009]

MECHANIK der Kontinua – Klausur 09 – Lösungen

TEIL A

1

(a) $\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$

(b) $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \xrightarrow[\text{for incompressible flow}]{\nabla\cdot} \vec{\nabla}\cdot\vec{u} = 0$

(c) $\rho \frac{dv_i}{dt} = \rho F_i - \nabla_i p + \eta \nabla_j \nabla_j v_i + (\eta + \eta')\nabla_i \nabla_j v_j$

for incompressible flow $\rightarrow \rho \frac{dv_i}{dt} = \rho F_i - \nabla_i p + \eta \nabla_j \nabla_j v_i$ Navier equat.

2

(i) For 2-D, incompressible, irrotational flows

(ii) $W(z) = \phi(z) + i\psi(z)$

(iii) $v_x = u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$; $v_y = v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

3 (i) $Re = \frac{vL}{\nu} = \frac{vL\rho}{\eta}$: v = velocity, L = width, ν = viscosity (kinematic), ρ = density, η = dynamic viscosity

$\left(Re = \frac{\text{viscous force}}{\text{inertia force}} \right)$

(ii) 2 flows are similar when their Reynold's numbers are the same: $Re_1 = Re_2$

(iii) $Re = \frac{uL}{\nu} = \frac{v\, d_{Rohr}\, \rho}{\eta} = 2000$ (laminar flow)

4 $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ll} + 2\mu \varepsilon_{ij}$: $\lambda, \mu \mathrel{\hat{=}}$ Lame constants

$\sigma_{ij} \mathrel{\hat{=}}$ stress tensor

$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial s_j}{\partial r_i} + \frac{\partial s_i}{\partial r_j} \right)$ $\varepsilon_{ij} \mathrel{\hat{=}}$ strain tensor velocity potential, velocity, pressure gradient

5 $\dot{\Phi} + \frac{v^2}{2} + U + \Pi = \text{const}$ — potential energy

valid along a streamline

for rotational flows

$\Phi \hat{=}$ velocity potential

$v \hat{=}$ velocity

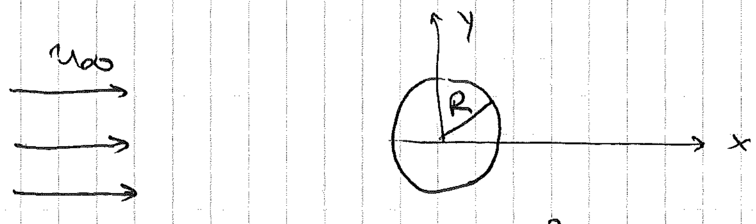
$U \hat{=}$ gradient of forces

$\Pi \hat{=}$ gratient of pressure

[In general Bernoulli's principle is equivalent to the principle of energy conservation (in a steady flow, the sum of all forms of mechanical energy in a fluid a streamline is the same at all points on that streamline]

TEIL B

6



$$W(z) = u_\infty\left(z + \frac{R^2}{z}\right) - i\frac{\Gamma}{2\pi}\ln z$$

(a) $z = re^{i\vartheta}$

$$\Rightarrow W(z) = u_\infty\left(re^{i\vartheta} + \frac{R^2}{re^{i\vartheta}}\right) - i\frac{\Gamma}{2\pi}\ln(re^{i\vartheta})$$

$$= u_\infty\left(re^{i\vartheta} + \frac{R^2}{r}e^{-i\vartheta}\right) - i\frac{\Gamma}{2\pi}\left[\ln r + i\vartheta\right]$$

$$= u_\infty\left(r(\cos\vartheta + i\sin\vartheta) + \frac{R^2}{r}(\cos\vartheta - i\sin\vartheta)\right) - i\frac{\Gamma}{2\pi}(\ln r + i\vartheta)$$

$$= \left[u_\infty r\cos\vartheta\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\vartheta\right] + i\left[u_\infty r\sin\vartheta\left(1 - \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\ln r\right]$$

$$= \phi + i\psi$$

$$\Rightarrow \begin{cases} \phi(r,\vartheta) = u_\infty r\cos\vartheta\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\vartheta \\ \psi(r,\vartheta) = u_\infty r\sin\vartheta\left(1 - \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\ln r \end{cases}$$

(b) $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\vartheta} = u_\infty\left[1 - \frac{R^2}{r^2}\right]\cos\vartheta$

$u_\vartheta = \frac{\partial\psi}{\partial r} - u_\infty\left[1 + \frac{R^2}{r^2}\right]\sin\vartheta + \frac{\Gamma}{2\pi r}$

(c) $r=R$: $u_\vartheta = \frac{\Gamma}{2\pi R} \equiv u_\infty \Rightarrow \Gamma = 2\pi u_\infty R$ (*)

$$\text{circulation} \triangleq \oint \vec{u}\cdot d\vec{\ell} = \oint (u_r\hat{r} + u_\vartheta\hat{\vartheta})\cdot R\,d\vartheta\,\hat{\vartheta} =$$

$$= \oint u_\vartheta R\,d\vartheta = R\int_0^{2\pi}\left[-u_\infty\left(1+\frac{R^2}{r^2}\right)\sin\vartheta + \frac{\Gamma}{2\pi R}\right]d\vartheta$$

$$= R\frac{\Gamma}{2\pi R}2\pi \overset{(*)}{=} \Gamma$$

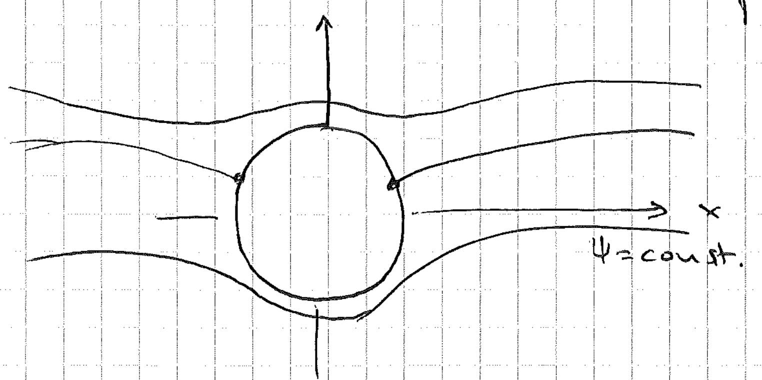
(d)

stagnation points: $u=0$ at $r=R$: $u_r(r=R)=0$

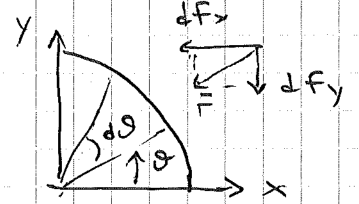
$$u_\vartheta(r=R) = -2u_\infty\sin\vartheta + \frac{\Gamma}{2\pi R}$$

$$\rightarrow u_\vartheta(r=R)=0 \Rightarrow \left.\begin{array}{l}\sin\vartheta_{sp} = \frac{\Gamma}{4\pi R u_\infty} \\ \Gamma = 2\pi u_\infty R\end{array}\right\} \rightarrow \sin\vartheta_{sp} = \frac{1}{2} \Rightarrow$$

$$\vartheta_{sp} = \frac{\pi}{6}, \frac{5\pi}{6}$$



(e)



$$dF_x = -pLR\cos\vartheta\,d\vartheta$$

$$dF_y = -pLR\sin\vartheta\,d\vartheta$$

but $p = p_\infty + c_p\frac{\rho}{2}u_\infty^2$

at $r=R$: $c_p = 1-\left[\frac{u_t}{u_\infty} - 2\sin\vartheta\right]^2$ $\quad u_t = u$ tangential

$$\Rightarrow F_x = -LR\int_0^{2\pi}\left\{\frac{\rho}{2}u_\infty^2\left[1-\left(\frac{u_t}{u_\infty}-2\sin\vartheta\right)^2\right]+p_\infty\right\}\cos\vartheta\,d\vartheta = 0$$

$$F_y = -LR\int_0^{2\pi}\left\{\frac{\rho}{2}u_\infty^2\left[1-\left(\frac{u_t}{u_\infty}-2\sin\vartheta\right)^2\right]+p_\infty\right\}\sin\vartheta\,d\vartheta =$$

$$= -2\pi\rho LR u_t u_\infty = -\rho u_\infty \Gamma L$$

or from Blasius formula:

$$F_x - iF_y = \frac{i\rho}{2}\oint_c\left(\frac{dw}{dz}\right)^2 dz$$

$$\frac{dw}{dz} = u_\infty \left(1 - \frac{R^2}{z^2}\right) - i\frac{\Gamma}{2\pi z}$$

$$\Rightarrow F_x - iF_y = \frac{1}{2} i\rho \oint_C \left[u_\infty \left(1 - \frac{R^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}\right]^2 dz =$$

$$= \frac{1}{2} i \rho \cdot 2\pi i \left(- \frac{i u_\infty \Gamma}{\pi}\right) = i\rho u_\infty \Gamma$$

$$\Rightarrow \begin{cases} F_x = 0 \\ F_y = \rho u_\infty \Gamma \end{cases}$$

(in the integral only $\frac{1}{z}$ term contributes with its residual $\mathrm{Res}\,(z=0)$)

$\boxed{7}$ (a) dynamic boundary condition at wall: $u = v_x(y=0) = 0$

and $\lim_{y \to \infty} v_x(y) = u_\infty$ or $v_x(y=\delta) = u_\infty$

at wall $v_y(y=0) = -v_a$

(b) $\rho = \text{const.} \longrightarrow \frac{D\rho}{Dt} = 0 \longrightarrow$ continuity eq.: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$

planar flow: $\frac{\partial}{\partial z} = 0$; also $\frac{\partial v_x}{\partial x} = 0 \Rightarrow \frac{\partial v_y}{\partial y} = 0$

$$\left.\begin{aligned} \Rightarrow v_y = v_y(x) &= \text{const} \\ v_y(y=0) &= -v_a \end{aligned}\right\} \Rightarrow v_y = -v_a = \text{const. throughout the flow}$$

(c) with $v_y = -v_a$, $\frac{\partial v_x}{\partial x} = 0$, $\frac{\partial v_z}{\partial z} = 0$, constant pressure, no body forces, steady flow $\Rightarrow$

$$-\rho v_a \frac{\partial v_x}{\partial y} = \eta \frac{\partial^2 u}{\partial y^2} \longrightarrow -v_a \frac{dv_x}{dy} = \underset{\text{viscosity}}{\nu} \frac{d^2 v_x}{dy^2} \Rightarrow$$

(solution)

$$\left.\begin{aligned} \Rightarrow v_x(y) &= C_1 + C_2 e^{-\frac{v_a}{\nu} y} \\ \text{boundary conditions} &\longrightarrow C_1 = -C_2 = u_\infty \end{aligned}\right\} \Rightarrow u_y(y) = u_\infty \left(1 - e^{-\frac{v_a}{\underset{\text{viscosity}}{\nu}} y}\right)$$

(d) Newtonian fluid (Cauchy – Poisson law):

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \varepsilon_{kk}\delta_{ij} + 2\eta\, \varepsilon_{ij}$$

only non-zero component on the plate surface : $\sigma_{xy}\big|_{y=0} = 2\eta\, \varepsilon_{xy}\big|_{y=0} = \eta \frac{\partial u_x}{\partial y}\big|_{y=0}$

from (c) : $\frac{\partial u_x}{\partial y}\big|_{y=0} = u_\infty \frac{V_a}{\nu} e^{-\frac{V_a y}{\nu}}\Big|_{y=0} = u_\infty \frac{V_a}{\nu}$ $\Rightarrow$

$$\Rightarrow \sigma_{xy}\big|_{y=0} = \rho u_\infty V_a$$

8

(a) Bernoulli eq: $p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2$

continuity eq: $v_1 A_1 = v_2 A_2 = v_3 A_3$ (*) $\Rightarrow$

$$v_2 = \sqrt{\frac{2\Delta p}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} = 12\,m/s$$

$\overset{(*)}{\Rightarrow}$ $v_1 = 4\,m/s$; $v_3 = 6\,m/s$

(b) Bernoulli eq: $p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$ $\Rightarrow$

$p_3 = p_a = 10^5\,N/m^2$ (outflow to the surrounding)

$\Rightarrow p_2 = 0{,}46 \cdot 10^5\,N/m^2$

similar for p_1 $\longrightarrow$ $p_1 = 1{,}1 \cdot 10^5\,N/m^2$

Bernoulli eq: $p + \rho g h = p_a + \frac{\rho}{2} v_3^2$

$$\Rightarrow p = 1{,}08 \cdot 10^5\,N/m^2$$

$\boxed{9}$ cylindrical symmetry $\rightarrow$ Navier Stokes (NS): $\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_\theta)}{dr}\right]=0$

(from ϑ component of NS eq in cylindrical coordinates with $u_r, u_z = 0$)

(a)

$$\Rightarrow u_\phi(r) = C_1 r + \frac{C_2}{r}$$

boundary conditions
$$\left\{\begin{array}{l} u_\phi(R_0) = \Omega_0 R_0 = C_1 R_0 + \frac{C_2}{R_0} \\ u_\phi(R_1) = \Omega_1 R_1 = C_1 R_1 + \frac{C_2}{R_1}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} C_1 = \dfrac{\Omega_0 R_0^2 - \Omega_1 R_1^2}{R_0^2 - R_1^2} \\ C_2 = \dfrac{(\Omega_1 - \Omega_0) R_1^2 R_0^2}{R_0^2 - R_1^2}\end{array}\right.$$

(a) from r-component of NS in cylindrical coordinates

$$\frac{dp}{dr} = \frac{\rho}{r} u_\vartheta^2 = \frac{\rho}{r}\left(C_1 r + \frac{C_2}{r}\right)^2$$

$$\Rightarrow p(r) = \frac{\rho}{2}\left(\frac{\Omega_0 R_0^2 - \Omega_1 R_1^2}{R_0^2 - R_1^2}\right) r^2 + 2\rho \frac{\Omega_0 R_0^2 - \Omega_1 R_1^2}{R_0^2 - R_1^2}\frac{(\Omega_1 - \Omega_0) R_1^2 R_0^2}{R_0^2 - R_1^2} \ln r$$

$$- \frac{\rho}{2}\left(\frac{(\Omega_1 - \Omega_0) R_1^2 R_0^2}{R_0^2 - R_1^2}\right)^2 \frac{1}{r^2} + \text{const.}$$

(in incompressible flow without pressure boundary conditions, the pressure can be determined only up to a constant)

(b) $$\vec{\nabla} \times \vec{u} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\vartheta) - \frac{\partial u_r}{\partial \vartheta}\right)\hat{e}_z = \frac{1}{r}\frac{d}{dr}\left(C_1 r^2 + C_2\right)\hat{e}_z$$

$$= 2 C_1 \hat{e}_z$$

for irrotational flow: $\vec{\nabla}\times\vec{u} = 0 \Rightarrow C_1 = 0 \Rightarrow$

$\Omega_0 R_0^2 = \Omega_1 R_1^2 \Rightarrow \dfrac{\Omega_0}{\Omega_1} = \left(\dfrac{R_1}{R_0}\right)^2$ viscous potential flow

$\boxed{10}$

$$\vec{s} = 5x\hat{e}_x + (y+2z)\hat{e}_y + (2y+z)\hat{e}_z$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial s_j}{\partial i} + \frac{\partial s_i}{\partial j}\right)$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial s_j}{\partial i} - \frac{\partial s_i}{\partial j}\right)$$

$$\sigma_{ij} = \lambda \varepsilon_{ee} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\Rightarrow \varepsilon_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{ij} = \begin{pmatrix} 7\lambda + 10\mu & 0 & 0 \\ 0 & 7\lambda + 2\mu & 4\mu \\ 0 & 4\mu & 7\lambda + 2\mu \end{pmatrix}$$

$$\det|\varepsilon_{ij} - \lambda\delta_{ij}| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0 \rightarrow$$

$$(5-\lambda)\left[(1-\lambda)^2 - 4\right] = 0 \Rightarrow \lambda_1 = 5, \ \lambda_2 = 3, \ \lambda_3 = -1$$

$$\varepsilon_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\lambda_1 = 5: \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow (x,y,z)_1 = (0,1,2)$$

$$\lambda_2 = 3 \longrightarrow (x,y,z)_2 = (0,1,1)$$

$$\lambda_3 = -1 \longrightarrow (x,y,z)_3 = (0,1,-1)$$