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# Solid State Theory
# Problem Set 9 — Elasticity

hand-out Fri 17.6., return Fri 24.6.

**1) Sound wave equation:** Starting from the Lagrange equation for the displacement field

$$\rho \ddot{u}_\alpha = -\frac{\delta \mathcal{F}}{\delta u_\alpha(\mathbf{r})},$$

derive the equation of motion for an isotropic solid. Use the expression for the free energy $\mathcal{F} = \sum_{\alpha\beta} \int d\mathbf{r} e_{\alpha\beta}\sigma_{\alpha\beta}/2$, stress tensor $\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\sum_\gamma e_{\gamma\gamma} + 2\mu e_{\alpha\beta}$, and strain tensor $e_{\alpha\beta} = (\partial u_\alpha/\partial r_\beta + \partial u_\beta/\partial r_\alpha)/2$. By calculating the divergence and the rotation of the equation of motion, obtain wave equations for longitudinal and transverse sound.

**2) Liquid crystals: Fredericksz transition** A nematic liquid crystal is composed of rod-like, top-down symmetric molecules. Within continuum elastic theory, it is described by the director unit-vector field $\hat{\mathbf{n}}(\mathbf{r})$ which determines the average molecular orientation within a volume element. Exploiting the top-down symmetry of molecules, $\hat{\mathbf{n}}(\mathbf{r}) \to -\hat{\mathbf{n}}(\mathbf{r})$, and translational/rotational symmetry of space, the linear elastic free energy reads

$$\mathcal{F} = \int d\mathbf{r} \left\{ K_1(\nabla \cdot \hat{\mathbf{n}})^2 + K_2(\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}))^2 + K_3(\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))^2 \right\}/2.$$

$K_1, K_2, K_3$ are the elastic coefficients describing, splay, twist, and bending.

a) Now consider a nematic between two walls which are parallel to the $y - z$ coordinates and at a separation $L$. At the walls, the nematic director shall be oriented along the z-direction. An applied electric tends to align liquid crystal molecules parallel to the field direction, described by the free energy term $\mathcal{F}_E = -\varepsilon_a \int d\mathbf{r}(\mathbf{E} \cdot \hat{\mathbf{n}})^2/2$. Let the electric field act along the y direction. Assume that the director field stays in the $y - z$ plane everywhere, i.e. $\hat{\mathbf{n}} = (0, \sin\theta(x), \cos\theta(x))$. Convince yourself that the electric field induces a twist deformation.

b) Derive the Euler-Lagrange equation describing the deformation. Perform the first integral. Use the boundary conditions at $x = 0$ and $x = L$ and the reflection symmetry condition at $x = L/2$. Calculate the deformation at the slab middle $x = L/2$ within an expansion for small deformation amplitude. What is the critical electric field strength below which no deformation is observed? How does the deformation grow slightly above the critical field strength?

Note that this switching behavior is essential to the functioning of liquid-crystal-display technology. (Hint: you need the integrals $\int_0^1 dx(1-x^2)^{-1/2} = \pi/2$ and $\int_0^1 dx x^2(1-x^2)^{-1/2} = \pi/4$ .)

**3) Gaussian expectation values:** Consider a scalar field theory $Z = \int \mathcal{D}\phi e^{-H[\phi]}$ described by the Gaussian action

$$H[\phi] = \int d^D r[g(\phi(r))^2 + b(\nabla\phi(r))^2]/2.$$

Calculate the Fourier transform of the correlation function

$$C(r-r') = \langle\phi(r)\phi(r')\rangle - \langle\phi(r)\rangle\langle\phi(r')\rangle.$$

a) Introduce a generating term $h(r)\phi(r)$ into the action which allows to calculate expectation values by taking appropriate derivatives of the partition function $Z$ with respect to $h(r)$.

b) Diagonalize the action by Fourier transforming, complete the square to obtain the $h$-dependence explicitly and exactly.

c) Calculate $\langle\tilde{\phi}(q)\tilde{\phi}(q')\rangle$ and from that $\tilde{C}(q)$.

d) Consider the mean-square fluctuation $C(0)$ for finite values of $g$ and $b$. For which dimensionality is $C(0)$ finite for i) infinite system size $L = \infty$ and finite lattice constant $a$ and ii) infinite system size $L = \infty$ and vanishing lattice constant $a = 0$?