T30f
Theoretische Teilchen- und Kernphysik
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Contents

1 Path Integral Quantization

1.1 Non-relativistic Quantum Mechanics

Let us start with a conceptual discussion, namely the *double slit* experiment (see Fig. 1.1). A particle emitted from a source S at time $t = 0$ passes through one or the other of the two holes A_1 and A_2 drilled in a screen, and it is detected at time $t = T$ by a detector located at O.

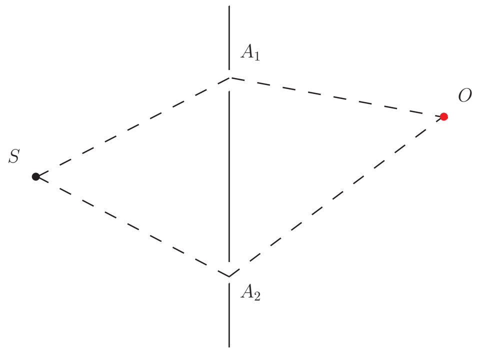


Figure 1.1: Double slit experiment.

According to the postulates of Quantum Mechanics (QM) we may say:

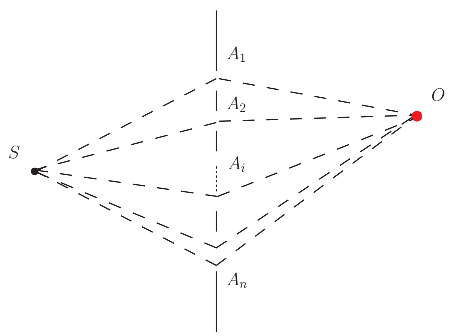
- The probability for the detection in O is given by

$$P(S \to O) = |A(S \to O)|^2 \,, \tag{1.1}$$

  where A is the amplitude of the process.

- From the superposition principle the amplitude for the detection in O is the sum of the amplitude for the particle to propagate from the source S through the hole A_1 and then onward to the point O, and the amplitude for the particle to propagate from S through A_2 and then to O:

$$A(S \to O) = A(S \to A_1 \to O) + A(S \to A_2 \to O) \,. \tag{1.2}$$



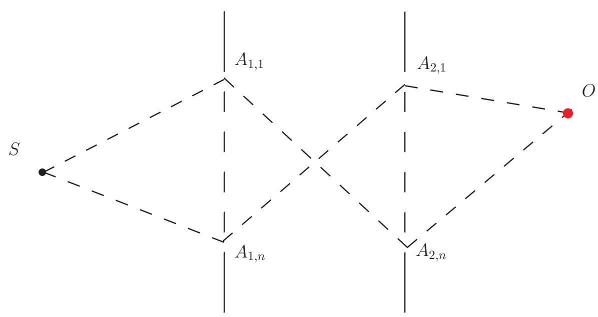


Figure 1.2: Experiment with one screen with n holes (left), experiment with two screens with n holes each (right).

We note that the corresponding probability contains the interference between the two terms in Eq. (1.2). If we drill n holes (see Fig. 1.2) we find for the amplitude the generalization of Eq. (1.2), that reads

$$A(S \to O) = \sum_{i=1}^{n} A(S \to A_i \to O)\,. \tag{1.3}$$

If we add a second screen with n holes (see again Fig. 1.2) we have for the amplitude

$$A(S \to O) = \sum_{i,j=1}^{n} A(S \to A_{1,i} \to A_{2,j} \to O)\,. \tag{1.4}$$

If one would add an infinite numbers of screens, drill infinite holes in each, we will get

$$A(S \to O) = \sum_{\text{all paths}} A(S \to \text{path} \to O)\,, \tag{1.5}$$

that is understood as a sum over all paths (see Fig. 1.3). Hence in order to calculate the amplitude we have to sum over the amplitude for the particle to propagate from the source to the detector following *all* possible paths.

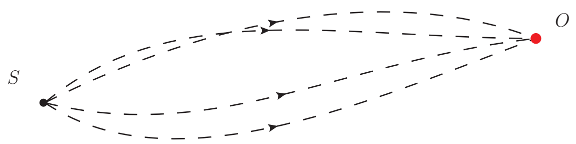
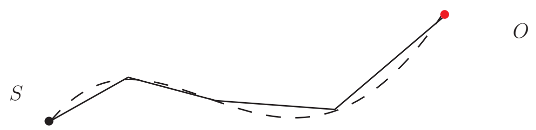


Figure 1.3: All possible paths in the limit of infinite holes and screens.

1) *How could one define the sum over the paths?*
   We have to consider the functional integral that is a sum over an infinite number of possible trajectories in order to compute a quantum amplitude. We can take a path and approximate it by straight line segments and let the segments go to zero.



2) *How can we construct the amplitude along a particular path?*
   If we know the amplitude for each infinitesimal segment, then we can just multiply them together to get the amplitude of the whole path.

In QM the amplitude to propagate from a point q_I to a point q_F in a time T is governed by the unitary operator $e^{-i\hat{H}T}$, where $\hat{H}$ is the Hamiltonian. The corresponding amplitude is given by $\langle q_F|e^{-i\hat{H}T}|q_I\rangle$.

1.2 Transition amplitude as path integrals

Feynman, following the ideas from Dirac, has shown that QM could be formulated in terms of path integrals. We discuss this approach to QM in details since it provides the key to the path integral formulation of QFT.

We consider a quantum mechanical system with one degree of freedom: one generalized coordinate, x, and its conjugate momentum, p. We keep $\hbar \neq 1$ and $c = 1$. In the canonical quantization we work with the Hilbert space operators $\hat{x}$ and $\hat{p}$ defined by the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar\,, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})\,. \tag{1.6}$$

where we also display the Hamiltonian of the system under consideration. The eigenstates of the position operator are introduced as:

$$\begin{cases} \hat{x}_H(t)|x,t\rangle_H = x|x,t\rangle_H\,, & \text{Heisenberg picture} \\ \hat{x}_S|x\rangle_S = x|x\rangle_S\,, & \text{Schrödinger picture} \end{cases} \tag{1.7}$$

with the relation between the two pictures that reads

$$\begin{cases} |x,t\rangle_H = e^{\frac{i\hat{H}t}{\hbar}}|x\rangle_S \\ |x\rangle_S = e^{-\frac{i\hat{H}t}{\hbar}}|x,t\rangle_H. \end{cases} \tag{1.8}$$

Since $\hat{x}_H(t)$ is time dependent, so are the eigenstates

$$\hat{x}_H(t) = e^{\frac{i\hat{H}t}{\hbar}}\,\hat{x}_S\,e^{-\frac{i\hat{H}t}{\hbar}}\,. \tag{1.9}$$

The eigenstates are normalized as follows:

$$\begin{cases} \langle x''|x'\rangle = \delta(x''-x')\,, & \int_{-\infty}^{+\infty} dx'|x'\rangle\langle x'| = 1\,, \\ \langle p''|p'\rangle = \delta(p''-p')\,, & \int_{-\infty}^{+\infty} dp'|p'\rangle\langle p'| = 1\,, \\ \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p'x'}{\hbar}}\,. \end{cases} \tag{1.10}$$

The matrix element ${}_H\langle x',t'|x,t\rangle_H = {}_S\langle x'|e^{-i\frac{\hat{H}(t'-t)}{\hbar}}|x\rangle_S$ corresponds to the transition from the eigenstate $|x,t\rangle$ at time t to the eigenstate $|x',t'\rangle$ at time t', hence it gives the *transition amplitude.*

This matrix element can be first expressed as a *multiple integral* which will then be used to define the *functional integral* (path integral) via a limiting procedure.

First we divide the time interval $(t'-t)$, characterizing the two states, into $(n+1)$ parts of equal lengths, ϵ, as follows

$$\begin{cases} (n+1)\epsilon = (t'-t) \\ t' = (n+1)\epsilon + t \\ t_j = j\epsilon + t\,, \quad j = 1,...,n\,. \end{cases} \tag{1.11}$$

and $x_0 = x$ for $t_0 = t$, and $x_{n+1} = x'$ for $t_{n+1} = t'$ are kept fixed. Then we can use the completeness relation at each of the time t_j:

$$\int dx_j\,|x_j,t_j\rangle_{HH}\langle x_j,t_j| = 1\,, \tag{1.12}$$

to derive

$$
\begin{aligned}
{}_H\langle x', t'|x, t\rangle_H &= \int dx_1 \cdots dx_n \; {}_H\langle x', t'|x_n, t_n\rangle_H \; {}_H\langle x_n, t_n|x_{n-1}\, , t_{n-1}\rangle_H \\
&\qquad \cdots {}_H\langle x_{j+1},\, t_{j+1}|x_j,\, t_j\rangle_H \cdots {}_H\langle x_1, t_1|x\, , t\rangle_H \qquad (1.13)
\end{aligned}
$$

Now we proceed using the following relation

$$
{}_H\langle x_j, t_j|x_{j-1}, t_{j-1}\rangle_H = \langle x_j|e^{-i\frac{\epsilon \hat{H}}{\hbar}}|x_{j-1}\rangle = \langle x_j|x_{j-1}\rangle - i\frac{\epsilon}{\hbar}\langle x_j|\hat{H}|x_{j-1}\rangle + \mathcal{O}(\epsilon^2)\, , \quad (1.14)
$$

where

$$
\begin{cases} x_0 = x,\; t_0 = t \\ x_{n+1} = x',\; t_{n+1} = t'. \end{cases} \qquad (1.15)
$$

are fixed. We choose a general form for the Hamiltonian (separable)

$$
\hat{H}(x, p) = \hat{f}(p) + \hat{g}(x), \qquad (1.16)
$$

so that we can write

$$
\langle x_j|\hat{H}|x_{j-1}\rangle = \int dp_j \langle x_j|p_j\rangle\langle p_j|\hat{H}|x_{j-1}\rangle = \int \frac{dp_j}{2\pi\hbar} exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1})\right] H(p_j, x_{j-1})\, , \qquad (1.17)
$$

and now $H(p_j, x_{j-1})$ is a classical number and not an operator any more. Inserting Eq. (1.17) inside Eq. (1.14) we obtain

$$
\begin{aligned}
{}_H\langle x_j, t_j|x_{j-1}, t_{j-1}\rangle_H &= \int \frac{dp_j}{2\pi\hbar}\, exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1})\right]\left[1 - \frac{i}{\hbar}\epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2) \\
&= \int \frac{dp_j}{2\pi\hbar}\, exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1}) - \frac{i}{\hbar}\epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2)\, , (1.18)
\end{aligned}
$$

and composing the infinitesimal transition amplitudes in (1.13) using eqs. (1.14) and (1.18) we find

$$
{}_H\langle x', t'|x, t\rangle_H = \lim_{n\to\infty} \int \prod_{j=1}^{n} dx_j \int \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} exp\left\{\frac{i}{\hbar}\sum_{j=1}^{n+1}\left[p_j(x_j - x_{j-1}) - H(p_j, x_{j-1})(t_j - t_{j-1})\right]\right\} \qquad (1.19)
$$

where the limit $n \to \infty$ ($\epsilon \to 0$) has been taken and terms of order $\mathcal{O}(\epsilon^2)$ neglected. So we obtain the transition amplitude as a path integral

$$
{}_H\langle x', t'|x, t\rangle_H = \int \mathcal{D}x\mathcal{D}p\, exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\, [p\dot{x} - H(p, x)]\right\}\, , \qquad (1.20)
$$

with $x(t) = x$ and $x(t') = x'$ fixed. The right-hand side in (1.20) is called functional integral over the phase space

$$
\mathcal{D}x = \prod_{j=1}^{n} dx_j\, , \quad \mathcal{D}p = \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} \qquad (1.21)
$$

in the limit $n \to \infty$.

If the Hamiltonian is of the simple form

$$
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})\, , \qquad (1.22)
$$

it is convenient to perform the p integration in (1.18). We define $\Delta x_j = x_j - x_{j-1}$ and we obtain

$$\int \frac{dp_j}{2\pi\hbar} exp\left\{\frac{i}{\hbar}\left(p_j\Delta x_j - \frac{p_j^2}{2m}\epsilon\right)\right\} = \frac{1}{2\pi\hbar}\sqrt{\frac{\pi 2m\hbar}{i\epsilon}} exp\left\{-\frac{(\Delta x_j)^2\hbar 2m}{4\hbar^2\epsilon i}\right\}$$
$$= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i\epsilon}} exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\Delta x_j}{\epsilon}\right)^2\epsilon\right\} = \frac{1}{C_j} exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\Delta x_j}{\epsilon}\right)^2\epsilon\right\}. \quad (1.23)$$

We used the result for the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + bx} = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{b^2}{4\alpha}}\,. \quad (1.24)$$

Such result can be equivalently obtained making the shift $p_j \to p_j + \frac{m\Delta x_j}{\epsilon}$, and the prefactor of the exponential in (1.23) reads

$$\frac{1}{C_j} = \int \frac{dp_j}{2\pi\hbar} exp\left\{-\frac{i}{\hbar}\frac{p_j^2}{2m}\epsilon\right\} = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i\epsilon}} \quad (1.25)$$

which is divergent in the limit $\epsilon \to 0$. However it is compensated by an analogous factor in the path integral. In this way the final result has the form of a functional integral over configuration space:

$${}_H\langle x', t'|x, t\rangle_H = \lim_{n\to\infty}\frac{1}{C}\int\prod_{j=1}^{n} dx_j \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x_j - x_{j-1})^2}{(t_j - t_{j-1})^2}(t_j - t_{j-1}) - V(x_{j-1})(t_j - t_{j-1})\right]\right\}, \quad (1.26)$$

which reads

$${}_H\langle x', t'|x, t\rangle_H = \frac{1}{C}\int \mathcal{D}x\, exp\left\{\frac{i}{\hbar}S\,[x]\right\}, \quad (1.27)$$

where $S\,[x] = \int_t^{t'} L(x, \dot{x})d\tau$ is the action integral over the trajectory $x(\tau)$ with the Lagrangian $L(x, \dot{x}) = m\dot{x}^2/2 - V(x)$. The normalization factor is given by

$$\frac{1}{C} = \prod_{j=1}^{N}\frac{1}{C_j} = \int \mathcal{D}p\, exp\left\{-\frac{i}{\hbar}\int_t^{t'}\frac{p^2}{2m}d\tau\right\}. \quad (1.28)$$

So we finally derived (1.27) *starting from a canonically quantized theory described by the Hamiltonian* (1.22). We can use another approach, define the quantum theory by the functional integral in (1.27), or in other words we can choose the path integral formulation as the quantization prescription for a system with the classical Hamiltonian in the form of (1.22). Then our derivation proves the equivalence of the *path integral* and *canonical quantization* methods for systems described by the Hamiltonian in (1.22).

From now on we will consider quantum theories defined by the path integral formulation. Let us consider

$${}_H\langle x', t'|x, t\rangle_H = \frac{1}{C}\int \mathcal{D}x\, exp\left\{\frac{i}{\hbar}S\,[x]\right\}, \quad (1.29)$$

if happens that

$$S\,[x(\tau)] \gg \hbar, \text{ i.e. } \hbar \to 0 \quad (1.30)$$

we can evaluate (1.29) by using the saddle point approximation. For $\hbar \to 0$ all the configurations far from the extrema of S give contributions that oscillate wildly under small deformations of the path and thus give zero contribution. The only first non-vanishing contribution comes from the extrema of the action,

$$\frac{\delta S}{\delta x} = 0\,, \tag{1.31}$$

which leads to the Euler–Lagrange equation

$$\frac{\partial L(x,\dot{x})}{\partial x} - \frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}} = 0\,. \tag{1.32}$$

This is the equation that defines the classical trajectory and the classical equations of motion in classical mechanics.

To obtain the semi-classical solution, one can use the power series expansion of the functional $S[x]$ about its classical solution

$$\begin{aligned} S[x] &= S[x_{\rm cl}] + S'[x](x - x_{\rm cl}) + \frac{1}{2}S''[x](x - x_{\rm cl})^2 + \mathcal{O}(x - x_{\rm cl})^3 \\ &= S[x_{\rm cl}] + \frac{1}{2}S''[x](x - x_{\rm cl})^2 + \mathcal{O}(x - x_{\rm cl})^3\,, \end{aligned} \tag{1.33}$$

where

$$\begin{aligned} S'[x] &= \frac{\delta S}{\delta x} \\ S''[x] &= \frac{\delta^2 S}{\delta x^2}\,. \end{aligned} \tag{1.34}$$

Truncating the above expansion and putting it in Eq. 1.29, one can see that

$$\frac{1}{C}\int \mathcal{D}x\,\exp\left\{\frac{i}{\hbar}S\,[x]\right\} \;\rightarrow\; \frac{1}{C}\exp\left\{\frac{i}{\hbar}S\,[x_{\rm cl}]\right\}\int \mathcal{D}x\,\exp\left\{\frac{i}{2\hbar}S''[x](x - x_{\rm cl})^2\right\}. \tag{1.35}$$

The term $\int \mathcal{D}x\,\exp\left\{\frac{i}{2\hbar}S''[x](x - x_{\rm cl})^2\right\}$ is the semi-classical contribution to the path integral. As a matter of fact, the path integral formulation allows a relatively simple understanding of the classical and semi-classical limit. As we increase $\hbar$ the classical trajectory still dominates but there are other paths close to it whose action is within $\Delta S \simeq \hbar$ and contribute significantly to the amplitude. The particle does an excursion around the classical trajectory.

The matrix element ${}_H\langle x', t'|x, t\rangle_H$ determines all the transition probabilities between quantum mechanical states. In view of the applications of the functional formalism to quantum field theories, it is important to know the path integral representation of the *matrix element of the position operators* that corresponds to the field operators of QFT.

For the time-ordered product of n such operators holds the formula

$${}_H\langle x', t'|T\hat{x}(t_1)\ldots\hat{x}(t_n)|x, t\rangle_H = \int \mathcal{D}x\mathcal{D}p\;x(t_1)\ldots x(t_n)\,e^{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{x} - H(p,x)]}\,. \tag{1.36}$$

Here the time-ordered product is defined as

$$T\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n) = \hat{x}(\tau_1)\hat{x}(\tau_2)\ldots\hat{x}(\tau_n), \tag{1.37}$$

where $\tau_i > \tau_{i+1}$ and $\tau_i = t_j$. We can show that (1.36) is true for the case of two operators $\hat{x}(\tau_1)\hat{x}(\tau_2)$ for $\tau_1 > \tau_2$. We divide again the time interval $(t' - t)$ into small intervals choosing $t_1 \dots t_n$ such that

$$\tau_1 = t_{i_1}\,, \quad \tau_2 = t_{i_2}\,, \tag{1.38}$$

and apply the completeness relation at each t_i. We have

$$\begin{aligned} {}_H\langle x', t'|\hat{x}(\tau_1)\hat{x}(\tau_2)|x, t\rangle_H &= \int \prod_i dx_i\ {}_H\langle x', t'|x_n, t_n\rangle_H \cdots {}_H\langle x_{i_1}, t_{i_1}|\hat{x}(\tau_1)|x_{i_1-1}\,, t_{i_1-1}\rangle_H \\ &\qquad \cdots {}_H\langle x_{i_2},\, t_{i_2}|\hat{x}(\tau_2)|x_{i_2-1},\, t_{i_2-1}\rangle_H \cdots {}_H\langle x_1, t_1|x\,, t\rangle_H \\ &= \int \prod_i dx_i x_{i_1} x_{i_2}\, {}_H\langle x', t'|x_n, t_n\rangle_H \cdots {}_H\langle x_1, t_1|x, t\rangle_H\,. \end{aligned} \tag{1.39}$$

Then if we proceed exactly as before we end up with (1.36). In particular, Eq. (1.39) is true when $\tau_1 > \tau_2$. When $\tau_2 > \tau_1$ the left-hand side of (1.39) is ${}_H\langle x', t'|\hat{x}(\tau_2)\hat{x}(\tau_1)|x, t\rangle_H$. So the path integral is equal to

$$\int \mathcal{D}x\mathcal{D}p\ x(\tau_1)x(\tau_2)\, e^{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{x} - H(p,x)]} = \begin{cases} {}_H\langle x', t'|\hat{x}(\tau_1)\hat{x}(\tau_2)|x, t\rangle_H\,,\ \tau_1 > \tau_2 \\ {}_H\langle x', t'|\hat{x}(\tau_2)\hat{x}(\tau_1)|x, t\rangle_H\,,\ \tau_2 > \tau_1\,. \end{cases} \tag{1.40}$$

As before it is possible to go from phase space path integrals to the path integrals over configuration space.

1.2.1 Transition amplitude in the presence of an external source $J(\tau)$

The transition amplitude with an external source reads

$${}_H\langle x', t'|x, t\rangle_H^J = \int \mathcal{D}x\mathcal{D}p\, exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\,[p\dot{x} - H(p, x) + \hbar J(\tau)x(\tau)]\right\}\,, \tag{1.41}$$

where we have modified the Hamiltonian with a source term $H \to H - \hbar Jx$. This transition amplitude in presence of an external source can be used as *generating functional* of the matrix element of the position operators. They are given by its functional derivatives with respect to $J(\tau)$:

$${}_H\langle x', t'|T\hat{x}(t_1)\dots\hat{x}(t_n)|x, t\rangle_H = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\, {}_H\langle x', t'|x, t\rangle_H^J\Bigg|_{J=0} \tag{1.42}$$

where the $\delta/\delta J(t)$ are the functional derivatives.

1.3 Vacuum to vacuum transitions in imaginary time formalism

In this section we do not explicitly indicate the Heisenberg vectors with the subscript H in order not to make the formulas too cumbersome. However all the bras and kets where both the position and the time appear are understood as bras and kets in the Heisenberg picture. In the case when the time appears in the bra and the ket, those are in the Heisenberg representation even if it is not declared with superscripts.

In QFT we are interested in the calculation of Green's functions that are matrix elements of field operators taken between *vacuum states*. It is then useful to consider the analogous problem in QM.
We assume that the Lagrangian of the system is time independent. The energy eigenstates are defined as $\hat{H}|n\rangle = E_n|n\rangle$. In configuration space (Schrödinger description) they correspond to the wave functions $\phi_n(x) = \langle x|n\rangle$. The ground state or vacuum state is described by the wave function $\phi_0(x) = \langle x|0\rangle$.
We have

$$\phi_0(t,x) = e^{-\frac{i}{\hbar}E_0 t}{}_S\langle x|0\rangle = \langle x|e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle = {}_H\langle x,t|0\rangle\,. \tag{1.43}$$

We are interested in $\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle$. It can be written, by inserting two completeness relations $\int dx'\,|x',t'\rangle\langle x',t'| = 1$ and $\int dx\,|x,t\rangle\langle x,t| = 1$, as follows

$$\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle = \int dx'\,dx\,\phi_0^*(t',x')\langle x',t'|T\hat{x}(t_1)\ldots\hat{x}(t_n)|x,t\rangle\phi_0(t,x) \tag{1.44}$$

and for the transition amplitude in (1.44) we can use the result given in (1.36). The vacuum expectation value written in (1.44) can also be obtained from a generating functional

$$\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)} Z[J]\Big|_{J=0} \tag{1.45}$$

with

$$Z[J] = \langle 0|0\rangle^J = \int dx'\,dx\,\phi_0^*(t',x')\langle x',t'|x,t\rangle^J\phi_0(t,x)\,. \tag{1.46}$$

One may understand the definition of the generating functional without the external current as follows

$$\langle 0|\int dx'\,|x',t'\rangle\langle x',t'|\int dx\,|x,t\rangle\langle x,t|0\rangle = \int dx'dx\phi_0^*(t',x')\phi_0(t,x)\langle x',t'|x,t\rangle = Z[0]\,, \tag{1.47}$$

then one can just replace $\langle x',t'|x,t\rangle$ with $\langle x',t'|x,t\rangle^J$ that is in turn given in (1.41).

It is very important to derive the generating functional $Z[J]$ in a different way. We are going to show that

$$Z[J] = \lim_{\substack{T_1\to+i\infty\\ T_2\to-i\infty}} \frac{exp\,[(i/\hbar)E_0(T_2-T_1)]}{\phi_0^*(x_1)\phi_0(x_2)}\langle x_2,T_2|x_1,T_1\rangle^J\,. \tag{1.48}$$

This implies that $Z[J]$ for $T_1\to+i\infty$ and $T_2\to-i\infty$ is determined by the transition amplitude $\langle x_2,T_2|x_1,T_1\rangle^J$ *at any given* x_1,x_2 (for example $x_1 = x_2 = 0$, it does not matter which values we choose for x_1 and x_2) provided that the analytic continuation to the imaginary values of T_1 and T_2 is done.

To prove eq. (1.48) we choose a source $J(\tau)$ that vanishes outside the time interval (t,t') with $T_2 > t' > t > T_1$. Then we can write

$$\langle x_2,T_2|x_1,T_1\rangle^J = \int dx'dx\langle x_2,T_2|x',t'\rangle\langle x',t'|x,t\rangle^J\langle x,t|x_1,T_1\rangle \tag{1.49}$$

where all the states are understood in the Heisenberg picture and we write

$${}_H\langle x,t|x_1,T_1\rangle_H = \langle x|e^{-\frac{i}{\hbar}H(t-T_1)}|x_1\rangle =$$

$$
\begin{aligned}
&= \sum_n \langle x|e^{-\frac{i}{\hbar}H(t-T_1)}|n\rangle\langle n|x_1\rangle \\
&= \sum_n \sum_{n'} \langle x|n'\rangle\langle n'|e^{-\frac{i}{\hbar}H(t-T_1)}|n\rangle\langle n|x_1\rangle = \\
&= \sum_n \langle x|n\rangle\langle n|x_1\rangle e^{-\frac{i}{\hbar}E_n(t-T_1)} \\
&= \sum_n \phi_n(x)\phi_n^*(x_1)e^{-\frac{i}{\hbar}E_n(t-T_1)} \longrightarrow \phi_0(x)\phi_0^*(x_1)e^{-\frac{i}{\hbar}E_0(t-T_1)} \qquad (1.50)
\end{aligned}
$$

where we have inserted two complete set of eigenstates to obtain the result. Similarly we can do the same for ${}_H\langle x_2, T_2|x', t'\rangle_H$. The only T-dependent terms is now the factor $e^{-\frac{i}{\hbar}E_n(t-T_1)}$ and we can continue $T \to +i\infty$ explicitly.

The oscillatory behaviour of the exponential is not well defined and we need to make some more precise statement in order to evaluate it unambiguously. To define it one can continue the time to imaginary values and then continue back to the real time at the end ($t = i\tau$).

In the limit $T \to \infty$ in the sum $\sum_n e^{-E_n T}$ only E_0 which is the lowest energy level survives. For this reason we can write, by using the result in (1.50) :

$$\lim_{T_1 \to +i\infty} e^{-\frac{i}{\hbar}E_0 T_1}\langle x, t|x_1, T_1\rangle = \phi_0(x)e^{-\frac{i}{\hbar}E_0 t}\phi_0^*(x_1) = \phi_0(t, x)\phi_0^*(x_1)\,. \qquad (1.51)$$

In the same way we calculate

$$\lim_{T_2 \to -i\infty} e^{\frac{i}{\hbar}E_0 T_2}\langle x_2, T_2|x', t'\rangle = \phi_0^*(t', x')\phi_0(x_2)\,. \qquad (1.52)$$

Then using (1.51) and (1.52) and the definition (1.46) in (1.49) we obtain

$$
\begin{aligned}
&\langle x_2, T_2|x_1, T_1\rangle^J = \int dx'dx\langle x_2, T_2|x', t'\rangle\langle x', t'|x, t\rangle^J\langle x, t|x_1, T_1\rangle \\
&= \int dx'dx e^{-\frac{i}{\hbar}E_0 T_2}\phi_0^*(t', x')\phi_0(x_2)\langle x', t'|x, t\rangle^J e^{\frac{i}{\hbar}E_0 T_1}\phi_0(x, t)\phi^*(x_1) \\
&= e^{-\frac{i}{\hbar}E_0(T_2 - T_1)}\phi_0(x_2)\phi^*(x_1)Z[J]\,. \qquad (1.53)
\end{aligned}
$$

We have therefore obtained (1.48) back and we have proven that $Z[J]$ can be given also in terms of (1.48).

Then if (1.48) is true we will also have

$$\langle 0|Tx(t_1)\ldots x(t_n)|0\rangle = \lim_{\substack{T_1 \to +i\infty \\ T_2 \to -i\infty}} \frac{e^{\frac{i}{\hbar}E_0(T_2 - T_1)}}{\phi_0^*(x_1)\phi_0(x_2)}\langle x_2, T_2|Tx(t_1)\cdots x(t_n)|x_1, T_1\rangle\,. \qquad (1.54)$$

Then the vacuum matrix elements can be calculated by taking functional derivatives of the generating functional $Z[J]$ given by (1.48). The J-independent factors are irrelevant because we can always consider quantities like

$$\frac{1}{\langle 0|0\rangle}\langle 0|Tx(t_1)\ldots x(t_n)|0\rangle = \left(\frac{1}{i}\right)^n \frac{1}{Z[0]}\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z[J]\bigg|_{J=0}\,. \qquad (1.55)$$

Then we can simply write in place of (1.48):

$$Z[J] = const \lim_{\substack{T_1 \to +i\infty \\ T_2 \to -i\infty}} \int \mathcal{D}x\, exp\left\{\frac{i}{\hbar}\int_{T_1}^{T_2}[L(x, \dot{x}) + \hbar Jx]\,dt\right\}\,, \qquad (1.56)$$

where x_1 and x_2 are arbitrary. The path integral is over all $x(t)$ satisfying the boundary conditions $\lim_{T_1\to i\infty} x(T_1) = x_1$ and $\lim_{T_2\to -i\infty} x(T_2) = x_2$. Moreover x_1 and x_2 are any chosen constants but often are taken to be zero.

1.4 Path integral formulation of Quantum Field Theory

Green's functions as path integrals

The results that we have obtained up to now are simple to generalize to the case of more than one degree of freedom. If *the number of degrees of freedom is to be d*, the coordinate x should be replaced by a d-component vector. The functional integral now corresponds to the sum over all trajectories in the d-dimensional configuration space, satisfying appropriate boundary conditions. For example, for $p \to \mathbf{p}$ and $x \to \mathbf{x}$ in $d = 3$ we have

$$\prod_{k=1}^{N+1} dp_k \to \prod_{k=1}^{N+1} dp_k^1 \prod_{k=1}^{N+1} dp_k^2 \prod_{k=1}^{N+1} dp_k^3 \,, \tag{1.57}$$

$$\prod_{h=1}^{N} dx_h \to \prod_{h=1}^{N} dx_h^1 \prod_{h=1}^{N} dx_h^2 \prod_{h=1}^{N} dx_h^3 \,. \tag{1.58}$$

In Field Theory, the trajectory $x(t)$ is replaced by a field function $\phi(t, \mathbf{x})$. The degrees of freedom are labelled by a continuous index $\mathbf{x}$, the number of degrees of freedom is *infinite*. In this case to define the appropriate path integral one can start from a *multiple* integral on a *discrete*, and for the moment *finite*, lattice of space-time points.

Here we need to insert the definition of the discretized generating functional. To this aim one divides the space into 4-dimensional cubes of volume ε^4 and identify each degree of freedom with discrete labels, that in turn enters the field and its derivative as follows

$$\phi_n \simeq \phi(t_l, x_i, y_j, z_k) \,, \tag{1.59}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{l,i,j,k} \simeq \frac{1}{\varepsilon} \left[\phi(t_l, x_i + \varepsilon, y_j, z_k) - \phi(t_l, x_i, y_j, z_k)\right] \,. \tag{1.60}$$

If then we put $n = l, i, j, k$ we can write

$$\mathcal{L}(\phi(t_l, x_i, y_j, z_k), \partial_\mu \phi(t_l, x_i, y_j, z_k)) = \mathcal{L}(\phi_n, \partial_\mu \phi_n) = \mathcal{L}_n \,, \tag{1.61}$$

and also for the generating functional

$$Z[J] = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \int \prod_{n}^{N^4} d\phi_n exp \left\{ i \sum_{n=1}^{N^4} \varepsilon^4 (\mathcal{L}_n + \hbar J_n \phi_n) \right\} \,. \tag{1.62}$$

This corresponds to defining QFT as a limit of a theory with only a finite number of degrees of freedom. The limit of an infinite lattice, which is related to the thermodynamical limit of statistical mechanics, already defines a theory with an infinite number of degrees of freedom.

However, this lattice theory does not have the usual Lorentz space-time invariance (it has a different symmetry in space-time) and one has to send the lattice step to zero and go to the continuous limit. This continuous limit is accompanied by infinities, the *ultraviolet* (UV) divergences of QFT (UV divergences $\to$ related to high energies/small distances). Then the definition of the functional integral in QFT is more ambiguous than in the case of quantum mechanics.

QFT is formulated in terms of vacuum expectation values of time-ordered products of field operators, the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \langle 0|T\phi(x_1)...\phi(x_n)|0\rangle \,. \tag{1.63}$$

We will write down the path integral representation for the $G^{(n)}$ using what we have developed up to here in QM. It is particularly important to remember the role played by Eqs. (1.48) and (1.54) in getting rid of the vacuum wave functions that are originally present in (1.44) as boundary conditions.

By analogy with the previous result we can postulate this path integral representation:

$$G^{(n)}(x_1, \dots, x_n) \sim \int \mathcal{D}\phi\, \phi(x_1)...\phi(x_n) e^{\frac{i}{\hbar}\int d^4x \mathcal{L}} \,, \tag{1.64}$$

where $\mathcal{D}\phi$ denotes the integration over all the functions $\phi(t, \mathbf{x})$ of space and time because for each value of $\mathbf{x}$, $\phi(t, \mathbf{x})$ corresponds to a separate degree of freedom. Moreover, the path integral is understood over all functions that satisfy the boundary conditions

$$\begin{cases} \phi(T_1, \mathbf{x}) = \phi_1(x)\,, \\ \phi(T_2, \mathbf{x}) = \phi_2(x)\,, \end{cases} \tag{1.65}$$

for $T_1 \to +i\infty$ and $T_2 \to -i\infty$, and $\phi_1(x)$ and $\phi_2(x)$ are arbitrary and can be put to zero. In fact (1.54) suggests that the boundary conditions are irrelevant.

In Eq. (1.64) we have an oscillatory behaviour. To make well defined calculations we can proceed in two ways:

1) Wick rotation: analytic continuation to imaginary time region, namely

$$\tau = -it\,, \quad t = i\tau\,. \tag{1.66}$$

In this way one obtains a particularly convenient path integral representation because the weight factor in the integral, $exp\,(-S_E/\hbar)$, is non-negative. Indeed we have that

$$S_E[\phi] = \int_E d^4x_E \mathcal{L}(x_E) = -i(-1)\int_M d^4x(-\mathcal{L}(x)) = -iS[\phi] \tag{1.67}$$

after we notice that $d^4x_E = -id^4x$ and

$$\mathcal{L}(x_E) = \frac{1}{2}\left[(\partial^E_\mu\phi)^2 + m^2\phi^2\right] = -\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) = -\mathcal{L}\,. \tag{1.68}$$

Then the Euclidean path integral formalism can be used to define Minkowski space Green's functions by analytic continuation of the Euclidean ones. So (1.64) may be understood as an analytic continuation in the variables $t_1, ..., t_n$ of the analogous Euclidean formula.

2) One can work in Minkowski space and regularize the integrand in the path integral by adding a small imaginary piece to the Lagrangian density $\mathcal{L}$ ($+i\epsilon$ prescription)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \frac{1}{2}i\epsilon\phi^2 \tag{1.69}$$

Eq. (1.64) has to be regarded as the formulation of the theory.

Now, what is the relation between this path integral definition and the usual canonical operator formulation of the QFT based on the same $\mathcal{L}$*?*

For our present scopes: they are equivalent if they originate the same perturbation theory and thus the same Feynman rules. We will check this in the following. The derivation of perturbation theory is much simpler in the functional framework, especially in the case of gauge field theories.

It is convenient to normalize the Green's functions by factorizing out the vacuum amplitude:

$$G^{(n)}(x_1, \dots, x_n) = \frac{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle}{\langle 0|0\rangle} = N\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n) exp\left\{\frac{i}{\hbar}\int d^4x\mathcal{L}\right\}\,, \tag{1.70}$$

where

$$\frac{1}{N} = \int \mathcal{D}\phi exp\left\{\frac{i}{\hbar}\int d^4x\mathcal{L}\right\} = \langle 0|0\rangle\,, \tag{1.71}$$

so that extra factors like those appearing in (1.54) are eliminated.

The Green's function in (1.70) are given *by the functional derivatives of the generating functional* $Z[J]$ *that gives the vacuum transition in the presence of an external sources*:

$$Z[J] = N\int \mathcal{D}\phi\, exp\left\{\frac{i}{\hbar}\int d^4x\left[\mathcal{L} + \hbar J(x)\phi(x)\right]\right\}\,, \tag{1.72}$$

and expanding $Z[J]$ in powers of J around $J = 0$ we obtain

$$Z[J] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int dx_1\cdots dx_n\, G^{(n)}(x_1, \dots, x_n)J(x_1)\cdots J(x_n) \tag{1.73}$$

and from this

$$G^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z[J]\bigg|_{J=0}\,. \tag{1.74}$$

The Green's functions can also be considered as the analytic continuation of those obtained from the generating functional defined in Euclidean space with $x_0 = ix_4$ ($t = i\tau$) with x_4 real:

$$Z_E[J] = N\int \mathcal{D}\phi\, exp\left\{-\frac{1}{\hbar}S_E[\phi(x)] + \int d^4x_E J(x)\phi(x)\right\}\,, \tag{1.75}$$

where $d^4x_E = dx_1\, dx_2\, dx_3\, dx_4$. For a scalar field

$$S_E[\phi(x_E)] = (1/2)\int d^4x_E\left[((\partial\phi)/(\partial x_4))^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{1.76}$$

we have

$$G_E^{(n)}(x_1 \dots x_n) = \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z_E[J]\bigg|_{J=0}\,. \tag{1.77}$$

Using the path integral formalism one can obtain the equations of motion (see exercise sheet No. 1), for example in free Euclidean scalar field theory.

Action quadratic in the fields

When the action is quadratic in the fields ($\hbar = 1$)

$$S = \frac{1}{2}\int d^4xd^4y\phi(x)A(x,y)\phi(y)\,, \tag{1.78}$$

the generating functional $Z[J]$ *can be calculated in a closed form.* Let us consider the example of the free Klein-Gordon theory that has an action quadratic in the fields. The Lagrangian density for the Klein-Gordon case is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)\,, \tag{1.79}$$

and

$$Z[J] = N\int \mathcal{D}\phi\, exp\left\{iS[\phi] + i\int d^4xJ(x)\phi(x)\right\}\,. \tag{1.80}$$

We write explicitly the exponent as follows

$$\int d^4x\,[\mathcal{L}(\phi) + J\phi] = \int d^4x\left[\frac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi + J\phi\right]\,, \tag{1.81}$$

where we have integrated by parts to obtain the right hand side of Eq. (1.81). This equation can be rewritten as

$$\int d^4x\,[\mathcal{L}(\phi) + J\phi] = \int d^4x\int d^4y\left[\frac{1}{2}\phi(x)(-\partial_x^2 - m^2 + i\epsilon)\delta(x-y)\phi(y) + \int d^4xJ(x)\phi(x)\right]\,, \tag{1.82}$$

where the subscript x in ∂_x^2 indicates that the partial derivatives are taken with respect to x. Comparing Eqs. (1.78) and (1.82), A can be identified as

$$A(x,y) = (-\partial_x^2 - m^2 + i\epsilon)\delta(x-y)\,. \tag{1.83}$$

Note that the prescription $+i\epsilon$ is introduced as a factor of convergence for the functional integral. We can complete the square by introducing a shifted field

$$\phi'(x) = \phi(x) - i\int d^4yD_F(x-y)J(y)\,, \tag{1.84}$$

where $D_F(x-y) = G^{(2)}(x-y)$ for the free particle and we shall exploit the relation

$$(\partial^2 + m^2 - i\epsilon)D_F(x-y) = -i\delta^{(4)}(x-y)\,. \tag{1.85}$$

Then we obtain

$$\begin{aligned}
&\int d^4x\,[\mathcal{L}(\phi) + J\phi] = \int d^4x\frac{1}{2}\left[\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] \\
&+i\int d^4x\int d^4y\,\frac{1}{2}\left[\phi'(-\partial_x^2 - m^2 + i\epsilon)D_F(x-y)J(y)\right] \\
&+i\int d^4x\int d^4y\,\frac{1}{2}\left[D_F(x-y)J(y)(-\partial_x^2 - m^2 + i\epsilon)\phi'(x)\right] \\
&+\frac{(i)^2}{2}\int d^4x\int d^4y\int d^4zD_F(x-y)J(y)(-\partial^2 - m^2 + i\epsilon)D_F(x-z)J(z)
\end{aligned}$$

$$+\int d^4x J(x)\phi'(x) + i\int d^4x \int d^4y J(x) D_F(x-y) J(y)$$
$$= \int d^4x \frac{1}{2}\left[\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] + \frac{i}{2}\int d^4x \int d^4y \phi'(x) i\delta^{(4)}(x-y) J(y)$$
$$+\frac{i}{2}\int d^4x \int d^4y i\delta^{(4)}(x-y) J(y)\phi'(x) - \frac{1}{2}\int d^4x \int d^4y \int d^4z D_F(x-y) J(y) i\delta^{(4)}(x-z) J(z)$$
$$+\int d^4x J(x)\phi'(x) + i\int d^4x \int d^4y J(x) D_F(x-y) J(y)$$
$$= \int d^4x \frac{1}{2}\left[\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] - \int d^4x J(x)\phi'(x) + \int d^4x J(x)\phi'(x)$$
$$-\frac{i}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y) + i\int d^4x \int d^4y J(x) D_F(x-y) J(y)$$
$$= \int d^4x \frac{1}{2}\left[\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] + \frac{i}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\,. \tag{1.86}$$

Now we change variable from ϕ to ϕ' in the functional integral in (1.80). Because this is a shift the Jacobian of the transformation is 1, so we obtain

$$Z[J] = N\int \mathcal{D}\phi' e^{i\int d^4\mathcal{L}(\phi')} \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\}$$
$$\Rightarrow Z[J] = \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\}\,. \tag{1.87}$$

To derive the above relation we used Eq. (1.71) to remove the normalization factor N. The functional generator is calculated in a closed form.

If we keep track of $\hbar$, we end up with

$$Z[J] = \exp\left\{-\frac{1}{2\hbar}\hbar^2\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\}\,. \tag{1.88}$$

In summary, *all the free Green's functions can be obtained via derivative of (1.87).*

We can obtain the result in Eq. (1.87) by using the results in the Appendix about Gaussian integrals. Indeed we can exploit the expressions in Eq. (D.27) to directly calculate the generator functional $Z[J]$ in the case of free action of the scalar field.

We start from

$$Z[J] = N\int \mathcal{D}\phi \exp\left\{i\int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi + J\phi\right]\right\}\,, \tag{1.89}$$

and we know that $(\partial_x^2 + m^2)D(x-y) = \delta^4(x-y)$ with $D(x-y) = iD_F(x-y)$. Then we have $D^{-1}(x-y) = (\partial_x^2 + m^2)\delta^{(4)}(x-y)$, which satisfies the definition

$$\int d^4z\, D^{-1}(x,z) D(z,y) = \delta^{(4)}(x-y). \tag{1.90}$$

This can be easily verified as $\int d^4z\, (\partial_x^2 + m^2)\delta^{(4)}(x-z) D(z,y) = (\partial_x^2 + m^2) D(x-y) = \delta^{(4)}(x-y)$. The integral in Eq. (1.89) is of the form of Eq. (D.28), with

$$A = -i(-\partial^2 - m^2 + i\epsilon)\,, \quad A^{-1} = \frac{i}{-\partial^2 - m^2 + i\epsilon} = -iD(x-y) = +D_F(x-y)\,. \tag{1.91}$$

So we can calculate

$$Z[J] = N\frac{1}{\sqrt{\det A}} \exp\left\{\frac{1}{2}\int d^4x \int d^4y iJ(x) D_F(x-y) iJ(y)\right\}$$

$$= N \frac{1}{\sqrt{\det A}} \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\} . \qquad (1.92)$$

Moreover, we can explicitly calculate the normalization factor

$$N^{-1} = \int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}}$$
$$= \int \mathcal{D}\phi \, \exp\left\{\frac{i}{2}\int d^4x \phi(-\partial^2 - m^2 + i\epsilon)\phi\right\} = \frac{1}{\sqrt{\det A}} . \qquad (1.93)$$

Then, it is easy to obtain

$$Z[J] = \frac{\sqrt{\det A}}{\sqrt{\det A}} \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\} , \qquad (1.94)$$

which coincides with (1.87) previously obtained with another method.

Notice

1) What is $\det A$, where $\det A = \det(-\partial^2 - m^2 + i\epsilon)$? It is an example of a functional determinant, and A here is a partial differential operator, which is the kernel of bilinear form that entered the action.

   We can study the eigenvalues of this operator for example in momentum space in Euclidean formulation. We have:

$$\det A \propto \prod_{k_E} (k_E^2 + m^2) \qquad (1.95)$$

   with $k_E^2 = k_4^2 + \mathbf{k}^2$ and $k_0 = ik_4$; this is a divergent quantity. However since this is a quantity that does not depend on J, we can cancel it by appropriately choosing the normalization as we have already done.

2) Eq. (1.94) shows that for the scalar free theory there is only one connected [1] Green's function: $D_F(x-y) = G^{(2)}(x-y)$. This is the Feynman propagator, it is represented in a Feynman diagram with a line. In general:

$$Z[J] = e^{\frac{i}{\hbar}W[J]} , \quad W[J] = -i(\hbar) \ln Z[J] \qquad (1.96)$$

   and

$$G^{(n)}_{\text{conn}}(x_1, \dots, x_n) = (-i)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)}\bigg|_{J=0} , \qquad (1.97)$$

   with

$$W[J] = (\hbar) \sum_{n=0}^{\infty} \frac{(i)^{n-1}}{n!} \int d^4x \dots d^4x_n G^{(n)}_{\text{conn}}(x_1, \dots, x_n) \, J(x_1) \dots J(x_n) . \qquad (1.98)$$

   W[J] is the generating functional of the connected Green's functions.

[1] Connected Green's functions can be represented by connected diagrams, *i.e.*, diagrams such that one can go throughout the diagram between any given two points of it.

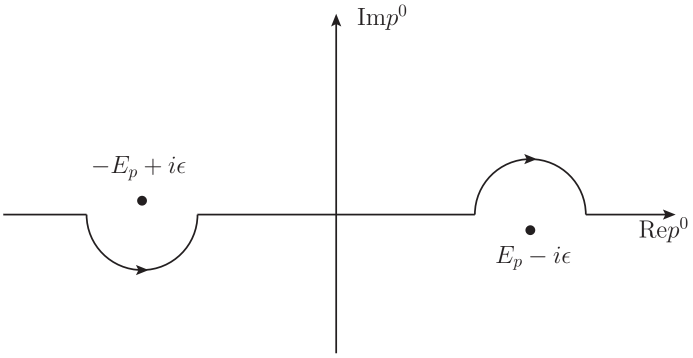


Figure 1.4: Poles of the free propagator.

3) The Feynman propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{1.99}$$

has been extensively studied in notes of the course *Relativity, Particles and Fields*, it also reads

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = G^{(2)}(x-y)\,. \tag{1.100}$$

We work on (1.99) to show it can be written as (1.100). The poles are

$$p_0^2 = \mathbf{p}^2 + m^2 - i\epsilon \Rightarrow p_0 = \pm E_p \mp i\epsilon\,, \tag{1.101}$$

which are displayed in Fig. 1.4.

For $x_0 > y_0$ we can close the contour in the lower half plane taking the contribution of the pole $E_\mathbf{p} - i\epsilon$. On the other hand, if $x_0 < y_0$ we can close the contour in the upper half plane taking the contribution of the pole $-E_\mathbf{p} + i\epsilon$. So we find

$$\begin{aligned} &\int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{(p_0 - E_\mathbf{p} + i\epsilon)(p_0 + E_\mathbf{p} - i\epsilon)} \\ &= -(2\pi i) \int \frac{d^3\mathbf{p}}{(2\pi)^4} \frac{i}{2E_\mathbf{p}} e^{-ip\cdot(x-y)}\Big|_{p_0 = E_\mathbf{p}} \theta(x_0 - y_0) \\ &\quad +(2\pi i) \int \frac{d^3\mathbf{p}}{(2\pi)^4} \frac{i}{-2E_\mathbf{p}} e^{-ip\cdot(x-y)}\Big|_{p_0 = -E_\mathbf{p}} \theta(y_0 - x_0)\,, \end{aligned} \tag{1.102}$$

where the minus sign in the first term of the right hand side of the equation arises because the corresponding contour integral is clockwise. Therefor for $x_0 > y_0$

$$\begin{aligned} D_F(x-y) &= -\frac{2\pi i}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i}{2E_\mathbf{p}} e^{-ip\cdot(x-y)} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} e^{-ip\cdot(x-y)} \\ &= \langle 0|\phi(x)\phi(y)|0\rangle\,, \end{aligned} \tag{1.103}$$

(For the last equality please see chapter 7 of notes of the course *Relativity, Particles and Fields.*) and for $x_0 < y_0$

$$D_F(x-y) = \frac{2\pi i}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i}{-2E_\mathbf{p}} e^{ip_0(x_0-y_0) - i\mathbf{p}(\mathbf{x}-\mathbf{y})} = +\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} e^{-ip\cdot(y-x)}$$

$$= \langle 0|\phi(y)\phi(x)|0\rangle \,, \tag{1.104}$$

so that

$$D_F(x-y) = \theta(x_0-y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0-x_0)\langle 0|\phi(y)\phi(0)|0\rangle$$
$$\equiv \langle 0|T\phi(x)\phi(y)|0\rangle \,. \tag{1.105}$$

2 Perturbation theory

2.1 Introduction

We consider the free scalar case. We can now prove the Wick's theorem [2] in the functional integration framework. Let us start with

$$G^{(n)}(x_1,\,\dots,x_n) = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots J(x_n)} Z[J]\Big|_{J=0}\,. \tag{2.1}$$

Then

$$\begin{aligned} G^{(2)}(x_1,x_2) &= (-i)^2 \frac{\delta^2}{\delta J(x_1)\delta J(x_2)} \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\}\Big|_{J=0} \\ &= D_F(x_1-x_2)\,. \end{aligned} \tag{2.2}$$

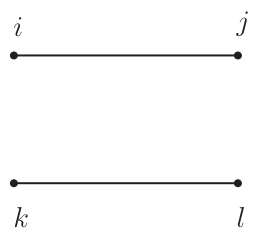
Then one can notice immediately that in this case all the Green's function with an odd number of points are zero.
Let us calculate the 4-point Green's function:

$$G^{(4)}(x_1,x_2,x_3,x_4) = (-i)^4 \frac{\delta^4}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \left(-\frac{1}{2}\right)^2 \frac{1}{2!} \sum_{i,j,k,\ell} J_i J_j J_k J_\ell D^F_{ij} D^F_{k\ell}\,, \tag{2.3}$$

where we have listed the only term in the series expansion of $Z[J]$ that gives a nonzero result and we find convenient to convert the integrals in the exponent of $Z[J]$ into sums over discrete points, with notations

$$J_i = J(x_i)\,, \quad D^F_{ij} = D_F(x_i - x_j)\,. \tag{2.4}$$

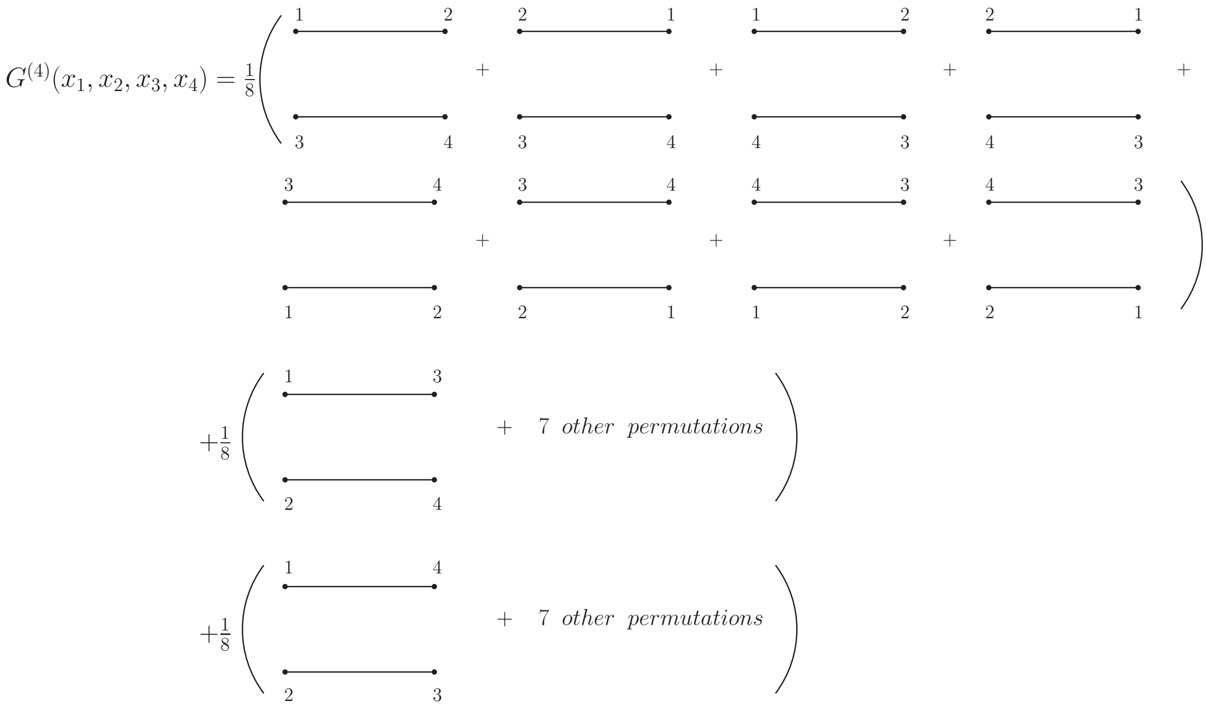
The structure is conveniently expressed in terms of diagrams of the type:



$$= \quad D^F_{ij}\; D^F_{kl} \quad . \tag{2.5}$$

[2] See also section 10.5 of notes of the course *Relativity, Particles and Fields.*

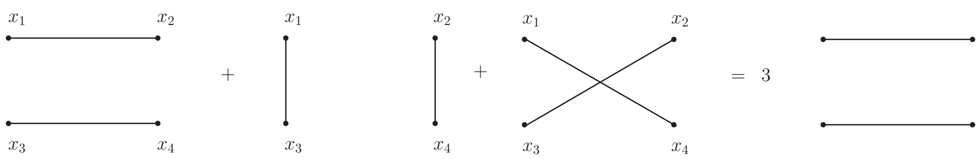
The derivative produce $4 \cdot 3 \cdot 2 = 24$ terms with the structure that can be represented diagrammatically as



or equivalently

$$
\begin{aligned}
G^{(4)}(x_1, x_2, x_3, x_4) = D_F(x_1 - x_2) D_F(x_3 - x_4) \;&+\; D_F(x_1 - x_3) D_F(x_2 - x_4) \\
&+\; D_F(x_1 - x_4) D_F(x_2 - x_3)\,, \quad (2.6)
\end{aligned}
$$

or since these diagrams are topologically equivalent we can also represent them as follows (Feynman diagrams in position space)



(2.7)

Eq. (2.6) is precisely the content of the Wick theorem that we have seen in the canonical quantization framework last semester. Indeed for the four-point function we have

$$
\begin{aligned}
& T\left\{\phi_1\phi_2\phi_3\phi_4\right\} =: \phi_1\phi_2\phi_3\phi_4 : + D_{12} : \phi_3\phi_4 : + \\
& D_{13} : \phi_2\phi_4 : + D_{14} : \phi_2\phi_3 : + D_{23} : \phi_1\phi_4 : + \\
& D_{24} : \phi_1\phi_3 : + D_{34} : \phi_1\phi_2 : + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}\,, \qquad (2.8)
\end{aligned}
$$

so that

$$
\langle 0 | T\left\{\phi_1\phi_2\phi_3\phi_4\right\} | 0 \rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}\,. \qquad (2.9)
$$

Momentum space Feynman rules

We can Fourier transform the source $J(x)$, then we have

$$J(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{J}(k) \,. \tag{2.10}$$

Recalling Eq. (1.89), the exponent of (free) $Z[J]$, which has a quadratic form, can be transformed into the Fourier space as:

$$\begin{aligned}
& -\frac{1}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \\
& = -\frac{1}{2} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \int d^4x \int d^4y \, i \frac{\tilde{J}(p_1) e^{-i(p_1+k)\cdot x} e^{-i(p_2-k)\cdot y} \tilde{J}(p_2)}{k^2 - m^2 + i\epsilon} \\
& = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} \,,
\end{aligned} \tag{2.11}$$

where we used

$$\int \frac{d^4x}{(2\pi)^4} e^{-i(p+k)\cdot x} = \delta^4(p+k) \tag{2.12}$$

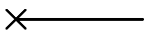
to evaluate the the integrals with respect to x and y, and in turn to get rid of the integrals with respect to p_1 and p_2.

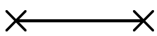
So we can write for the scalar free case

$$Z_{\text{free}}[J] = 1 - \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} + \frac{1}{2!} \left[-\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} \right]^2 + \cdots \,,$$

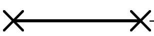
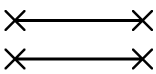
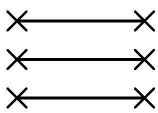
with the corresponding Feynman diagrams (free propagation, source, propagation between two sources respectively) as follows

 $= \dfrac{i}{k^2 - m^2 + i\epsilon}$

J  $= iJ(k)$

J  $J \; = -i \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} \,.$ (2.13)

So that we can write graphically the functional generator for the free scalar case:

$Z[J] = 1 + \frac{1}{2}$  $+ \frac{1}{2!}\left(\frac{1}{2}\right)^2$  $+ \frac{1}{3!}\left(\frac{1}{2}\right)^3$  $+ \cdots$ (2.14)

2.2 Perturbation theory: interacting case $\lambda\phi^4$

Let us recall the Lagrangian for the $\lambda\phi^4$ interacting theory that reads

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4 \,, \tag{2.15}$$

and we want to calculate Green's functions in this theory.
The generator functional is now

$$Z[J] = N \int \mathcal{D}\phi \, e^{i\int d^4x(\mathcal{L}+J\phi)} \,, \tag{2.16}$$

normalized so that $Z[0] = 1$ and with $\mathcal{L}$ given by (2.15). In this case we can no longer calculate exactly $Z[J]$. To calculate the Green's functions we have to use a perturbative expansion defined in terms of the interaction term:

$$S_I = S - S_0 \,, \quad S_0 = S_{\text{free}} \,. \tag{2.17}$$

S_I is regarded as small; typically the interaction term is weighted by a small coupling constant like the λ in $\lambda\phi^4/4!$. Then we can write the identity:

$$\begin{aligned}
&\int \mathcal{D}\phi \, exp\left\{ i\left(S_0[\phi] + S_I[\phi] + \int d^4x J(x)\phi(x) \right) \right\} \\
&\equiv \exp\left\{ iS_I\left[\frac{1}{i}\frac{\delta}{\delta J} \right] \right\} \int \mathcal{D}\phi \, \exp\left\{ i\left(S_0[\phi] + \int d^4x J(x)\phi(x) \right) \right\} \\
&= \exp\left\{ iS_I\left[\frac{1}{i}\frac{\delta}{\delta J} \right] \right\} Z_0[J] \,,
\end{aligned} \tag{2.18}$$

where $Z_0[J] = Z_{\text{free}}[J]$ is the functional generator for the free case that we are able to calculate in a closed form (we are going to use $Z_{\text{free}}[J]$ in the following).

The perturbative series is generated by expanding the exponential factor $\exp(iS_I[\delta/i\delta J])$ in powers of S_I and performing the functional derivatives as indicated.
This is equivalent to expanding $\exp(iS_I[\phi])$ under the path integral. So we obtain in perturbation theory this general formula for the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \frac{\int \mathcal{D}\phi \, \phi(x_1)\cdots\phi(x_n) \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(iS_I[\phi] \right)^m \right] e^{iS_0[\phi]}}{\int \mathcal{D}\phi \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(iS_I[\phi] \right)^m \right] e^{iS_0[\phi]}} \,. \tag{2.19}$$

Let us now summarize the interaction Lagrangian, action, and the generating functional for the $\lambda\phi^4/4!$ theory as follows:

$$\begin{cases} \mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 \,, \\ S_I = \int d^4x \left[-\frac{\lambda}{4!}\phi^4 \right] \,, \end{cases} \tag{2.20}$$

$$\begin{cases} Z[J] = N \, \exp\left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} Z_{\text{free}}[J] \,, \\ \frac{1}{N} = \exp\left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} Z_{\text{free}}[J] \mid_{J=0} \,, \\ Z_{\text{free}}[J] = \exp\left\{ -\frac{1}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \right\} \,. \end{cases} \tag{2.21}$$

We consider a perturbative expansion in a small coupling constant λ so that

$$\exp\left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} \simeq 1 - i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 + \cdots \tag{2.22}$$

Let us calculate the following quantities

$$\frac{\delta}{i\delta J(y)} Z_{\text{free}}[J] = \left\{ +i \int d^4x D_F(x-y) J(x) \right\} Z_{\text{free}}[J]\,, \tag{2.23}$$

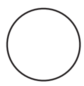
$$\left(\frac{\delta}{i\delta J(y)}\right)^2 Z_{\text{free}}[J] = \left\{ D_F(0) - \left(\int d^4x D_F(x-y) J(x) \right)^2 \right\} Z_{\text{free}}[J]\,, \tag{2.24}$$

$$\begin{aligned}
\left(\frac{\delta}{i\delta J(y)}\right)^3 Z_{\text{free}}[J] &= \Big\{ iD_F(0) \left(\int d^4x D_F(x-y) J(x) \right) \\
&+2iD_F(0) \left(\int d^4x D_F(x-y) J(x) \right) - i \left(\int d^4x D_F(x-y) J(x) \right)^3 \Big\} Z_{\text{free}}[J] \\
&= \left\{ 3iD_F(0) \left(\int d^4x D_F(x-y) J(x) \right) - i \left(\int d^4x D_F(x-y) J(x) \right)^3 \right\} Z_{\text{free}}[J]\,,
\end{aligned} \tag{2.25}$$

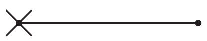
$$\begin{aligned}
\left(\frac{\delta}{i\delta J(y)}\right)^4 Z_{\text{free}}[J] &= \Big\{ 3(D_F(0))^2 - 6D_F(0) \left(\int d^4x D_F(x-y) J(x) \right)^2 \\
&+ \left(\int d^4x D_F(x-y) J(x) \right)^4 \Big\} Z_{\text{free}}[J]\,.
\end{aligned} \tag{2.26}$$

We can use Feynman diagrams in position space to represent those formulas and we list them as follows

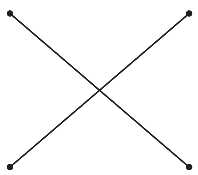
$D_F(x-y)$



$D_F(0)$



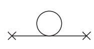
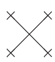
$J(x)$



$-i\lambda$

.

Then we can write

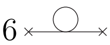
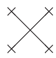
$\lambda \left(\frac{\delta}{i\delta J(y)}\right)^4 Z_{\text{free}}[J] = \lambda \Big\{ 3$  $- 6$  $+$  $\Big\} Z_{\text{free}}[J]\,.$ (2.27)

Then we can calculate also at this same order the normalization factor

$$
\begin{aligned}
N^{-1} &= \exp\left\{-i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4\right\} Z_{\text{free}}[J]\Bigg|_{J=0} \\
&= \left(1 - i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right) Z_{\text{free}}[J]\Bigg|_{J=0} \\
&= 1 - i\frac{\lambda}{4!}\int d^4y\, 3(D_F(0))^2\,, \qquad (2.28)
\end{aligned}
$$

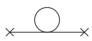
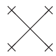
where in the last step we used (2.27) and put $J = 0$. So at the first order in the λ expansion we have

$$
Z[J] = \frac{\left\{1 - i\frac{\lambda}{4!}\int d^4y\left[3 \; \text{(diagram)} \; - 6 \; \text{(diagram)} \; + \; \text{(diagram)}\right]\right\} Z_{\text{free}}[J]}{\left[1 - i\frac{\lambda}{4!}\int d^4y\,(3\; \text{(diagram)})\right]}\,. \qquad (2.29)
$$


We see already that the normalization N contains vacuum diagrams that cancel with the same vacuum diagrams contribution in $Z[J]$, at this order it is the diagram .
Then since the vacuum diagrams (vacuum fluctuations) cancel between the numerator and the denominator we are left with

$$
Z[J] = 1 - i\frac{\lambda}{4!}\int d^4y\left[-6 \; \text{(diagram)} \; + \; \text{(diagram)}\right] Z_{\text{free}}[J] + \mathcal{O}(\lambda^2)\,. \qquad (2.30)
$$

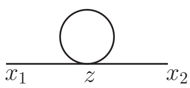



The cancellation of the vacuum diagrams occurs at any order in perturbation theory and it is a property of the normalized $Z[J]$.

Calculation of Green's functions in the $\lambda\phi^4/4!$ theory at order λ: $G^{(2)}$ and $G^{(4)}$

Let us start with the two point Green's function

$$
\begin{aligned}
G^{(2)}(x_1, x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J(x_1)\delta J(x_2)} Z[J]\Bigg|_{J=0} \\
&= (-1)\frac{\delta}{\delta J(x_1)\delta J(x_2)}\left[\left(1 - i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4\right) Z_{\text{free}}[J]\right]_{J=0} \\
&= D_F(x_1 - x_2) - i\frac{\lambda}{4!} 2\cdot 6\, D_F(0)\int d^4z D_F(z - x_1) D_F(z - x_2) + \mathcal{O}(\lambda^2)
\end{aligned}
$$

$= $  $- \frac{i\lambda}{2}$  . (2.31)

Up to order λ the effect of the interaction on the free particle propagator is described by the last diagram in (2.31). But what is the physical result of this interaction? We see now that the effect of the interaction is to change the value of the physical mass away from m.
We know that

$$
D_F(x - y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \qquad (2.32)
$$

has poles at $p^2 = \pm m$. Now let us evaluate the following quantity:

$$
\begin{aligned}
& -i\frac{\lambda}{2}D_F(0)\int d^4z D_F(z-x_1)D_F(z-x_2) \\
&= -i\frac{\lambda}{2}D_F(0)\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int d^4z \frac{ie^{-ip_1\cdot(z-x_1)}}{p_1^2-m^2+i\epsilon}\frac{ie^{-ip_2\cdot(z-x_2)}}{p_2^2-m^2+i\epsilon} \\
&= +i\frac{\lambda}{2}D_F(0)\int \frac{d^4p_1}{(2\pi)^4}\int d^4p_2\delta^4(p_1+p_2)\frac{e^{ip_1\cdot x_1+ip_2\cdot x_2}}{(p_1^2-m^2+i\epsilon)(p_2^2-m^2+i\epsilon)} \\
&= +i\frac{\lambda}{2}D_F(0)\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2}\,, \qquad (2.33)
\end{aligned}
$$

where we called $p_1 = p$. Therefore we obtain for the Green's function:

$$
G^{(2)}(x_1,x_2) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\epsilon}\left(1+\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}\right)+\mathcal{O}(\lambda^2)\,. \qquad (2.34)
$$

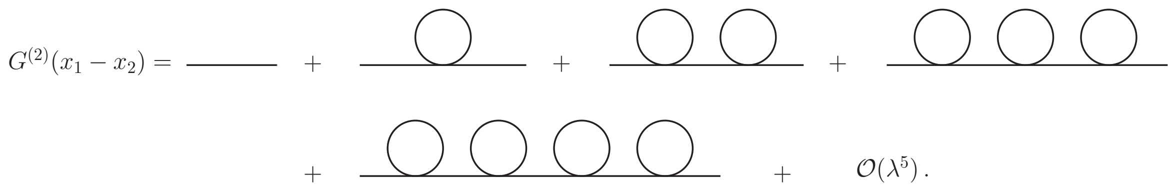
Formally we can rewrite

$$
1+\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}+\mathcal{O}(\lambda^2) = \left(\frac{1}{1-\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}}\right)\,, \qquad (2.35)
$$

the last equality is valid up to $\mathcal{O}(\lambda^2)$. We have used

$$
\frac{1}{1-x} = 1+x+\mathcal{O}(x^2)\,, \qquad (2.36)
$$

and this series resummation corresponds to taking into account all the diagrams of the *tadpole* type



Using (2.34) and (2.35) we obtain

$$
\begin{aligned}
G^{(2)}(x_1,x_2) &= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\epsilon}\left(\frac{p^2-m^2+i\epsilon}{p^2-m^2+i\epsilon-\frac{\lambda}{2}D_F(0)}\right) \\
&= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2-\frac{\lambda}{2}D_F(0)+i\epsilon}\,. \qquad (2.37)
\end{aligned}
$$

So we see that the Fourier transform of $G^{(2)}(x_1,x_2)$ now has a pole at $p^2 = m^2 + (\lambda/2)D_F(0) \equiv m^2+\delta m^2 = m_R^2$. Hence *the particle that propagates in an interacting theory* has a physical mass equal to m_R (renormalized) which is the mass that can be measured. The mass m in the Lagrangian is not the same as the physical mass m_R in an interacting theory, and m can not be measured. The change in mass is related to the self-interacting term in the Lagrangian, namely $(\lambda/4!)\phi^4$.
The change in the mass, δm^2, is quadratically divergent:

$$
D_F(0) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} \sim \frac{p^4}{p^2}\,, \qquad (2.38)
$$

indeed there are four powers of p in the numerator and two in the denominator. One can also see it by putting a cutoff in the above integral. The mass is changed by an infinite constant. The divergence is UV and it comes from very large momenta or very small space distances. We will deal with those infinities in the renormalization of $(\lambda/4!)\phi^4$ theory.

Notice: Power counting.
A way to evaluate the degree of divergence of the integrals of the type in eq. (2.38) is the power counting, i.e. to count the powers in p at the numerator and denominator and subtract them. In the case of eq. (2.38) at the numerator the momentum appears with exponent 4 and at the denominator appears with exponent 2, so that the divergence is $D = 4 - 2 = 2$ (quadratic divergence).

Let us now discuss the four point Green's function.

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) &= \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \\ &= \left.\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}, \end{aligned} \tag{2.39}$$

for the free case we have already obtain it in eq. (2.6), that we repeat here

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) = D_F(x_1 - x_2)D_F(x_3 - x_4) &+ D_F(x_1 - x_3)D_F(x_2 - x_4) \\ &+ D_F(x_1 - x_4)D_F(x_2 - x_3)\,. \end{aligned} \tag{2.40}$$

We are interested only in the order λ of th expansion of $Z[J]$:

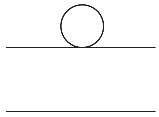
$$\begin{aligned} Z[J] = N &\left[1 - i\frac{\lambda}{4!}\int d^4z \left(\frac{\delta}{i\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right] \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\} \\ = N &\left[1 - \int d^4z \left(\frac{i\lambda}{4!}\right) 3(D_F(0))^2 + \frac{i\lambda}{4!} 6D_F(0)\int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^2 \right. \\ &\left. -\frac{i\lambda}{4!}\int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^4\right] \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\} \\ &+\mathcal{O}(\lambda^2)\,. \end{aligned} \tag{2.41}$$

Note that the above relation is identical to eq. 2.30 after substituting the normalization factor N. Now, one can see that:

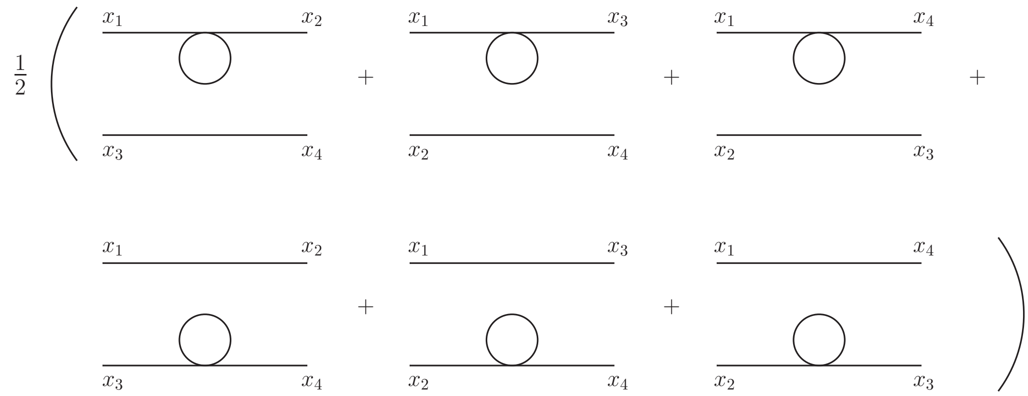
- The 1 in eq. (2.41) gives the terms in (2.40) when one applies the functional derivative with the four currents;
- the second term, in principle, gives a multiple of the vacuum diagram, but it does not contribute since it will be canceled out by the renormalization factor N;
- the third term in (2.41) gives, after derivation:

$$\begin{aligned} -i\frac{\lambda}{2}D_F(0)\int d^4z \quad &[D_F(z - x_1)D_F(z - x_2)D_F(x_3 - x_4) \\ &+D_F(z - x_1)D_F(z - x_3)D_F(x_2 - x_4) \\ &+D_F(z - x_1)D_F(z - x_4)D_F(x_2 - x_3) \end{aligned}$$

$$+D_F(z-x_2)D_F(z-x_3)D_F(x_1-x_4)$$
$$+D_F(z-x_2)D_F(z-x_4)D_F(x_1-x_3)$$
$$+\, D_F(z-x_3)D_F(z-x_4)D_F(x_1-x_2)]$$

$= -3i\lambda$ () , (2.42)

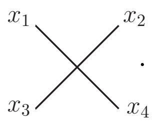
and the diagrams in the last line of (2.42) has to be understood as



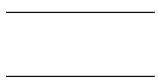
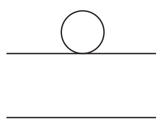
, (2.43)

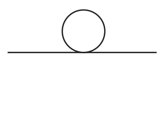
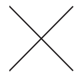
- the fourth term in (2.41) gives, after derivation:

$$-i\lambda \int d^4z D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)$$

$= -i\lambda$  . (2.44)

We can then write:

$G^{(4)}(x_1,x_2,x_3,x_4) = 3$ () $- 3i\lambda$ () $- i\lambda$ ()

$= 3$ ( ) $- i\dfrac{\lambda}{4!}\Big[12\times 6$ () $+ 24$ () $\Big]$. (2.45)

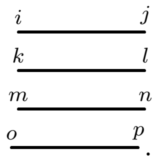
The numerical coefficients in (2.45) come from pure combinatorics and they are called weight factors. Let us see how it works. The contributing term in the $G^{(4)}$ is

$$-i\frac{\lambda}{4!}\int d^4z\, \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}\left(\frac{\delta}{\delta J(z)}\right)^4 Z[J]_{\rm free}\Bigg|_{J=0}\,. \tag{2.46}$$

The only non zero term comes from the term in the expansion of $Z[J]_{\rm free}$ that contains four propagators:

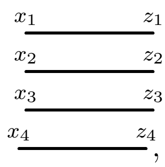
$$\left(-\frac{1}{2}\right)^4\frac{1}{4!}\left[\int d^4x\int d^4y J(x)D_F(x-y)J(y)\right]^4\,. \tag{2.47}$$

So at the end we will have four propagators

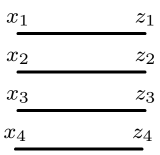
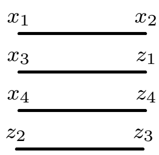
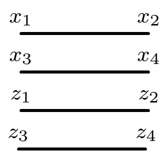


The differentiation over (2.47) provides all possible assignments of points x_1, x_2, x_3, x_4 and four points z (that it is convenient to split in z_1, z_2, z_3, z_4 and remembering at the end that $z_1 = z_2 = z_3 = z_4$) to the points i, j, k, l, m, n, o, p. Let us discuss the several contributions.
First let us assume that the pairs of points joined by the propagators are specified, e.g. $x_1 z_1$, $x_2 z_2$, $x_3 z_3$ and $x_4 z_4$. Then we have

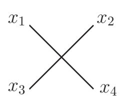
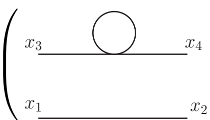
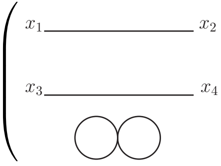
- an additional combinatorial factor following from the "horizontal" symmetry (exchange $x_1 \leftrightarrow z_1$) which is 2^n for n pairs,
- another combinatorial factor coming from the "vertical" symmetry, e.g. exchanging the pairs $x_1 - z_1$ with $x_2 - z_2$ in



which is $n!$ for n pairs. These factors cancel with the $1/(2^n n!)$ present in (2.47) and this happens for any term of the expansion contributing to the Green's functions. We have to remember the factor $1/N!$ that comes from the series expansion $exp(iS_I) = \sum_{N=0}^{\infty}(i)^N/N!(S_I)^N$. For the case $\mathcal{O}(\lambda)$ we have $N = 1$ and we should remember the factor $1/4!$ present in the definition of the interaction $\lambda/4!\phi^4$.
Finally we should take into account all the possible choices of pairs of points x_i and z_i joined by the propagator. One obtains

$4!$  $+ 3!\, 4 \times 3$  $+ 3 \times 3$  .

Then for $z_1, z_2, z_3, z_4 \to z$ we get

$-i\frac{\lambda}{4!}\Bigg\{ 4! \Bigg($  $\Bigg) + 4! \times 3 \Bigg($  $\Bigg) + 3 \times 3 \Bigg($  $\Bigg)\Bigg\}$, (2.48)

We have three topologically distinct type of diagrams and we can understand the weight factors in front of each class in the following way:

1) $-$ x_1 connected to z_1 or z_2 or z_3 or z_4: 4 ways,

- x_2 connected to the three z remained: 3 ways,
- x_3 connected to the two z remained: 2 ways,
- x_4 connected to the one z remained: 1 ways,

2) – x_1 connected to x_2 or x_3 or x_4: 3 ways,
   – z_i connected to the two x remained: 4×2 ways,
   – remained three z_i connected to the one x remained: 3 ways,

3) – x_1 connected with x_2 or x_3 or x_4: 3 ways,
   – z_1 connected with the three z remained: 3 ways.

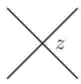
Remember that the analytic expression for the first diagram (vertex diagram) in Eq. 2.48 is

$$-i\lambda \int d^4z D_F(x_1 - z) D_F(x_2 - z) D_F(x_3 - z) D_F(x_4 - z)\,. \tag{2.49}$$

In summary, we can arrive at the following Feynman rules for $\lambda/4!\phi^4$ in coordinate space:

propagator line 

$= \quad D_F(x-y)$

vertex 

$= \quad -i\lambda$ with integration over z

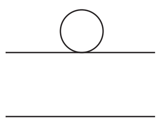
divide by the symmetry factor S (2.50)

It is important to remark here that dividing by the symmetry factor S is equivalent to multiplying by the weight of the diagram.

Notice

1) in the calculation of realistic processes involving e. g. electrons and photons, the particles are not identical and there are not symmetry factors.

2) the diagrams like



(2.51)

in $G^{(4)}$ only contribute to the trivial (diagonal) part of the S matrix so it is not relevant. It describes two particles moving independently and the effect of the interaction is to modify the propagator of one of them. This graph is called *disconnected.* The other graph of order λ,

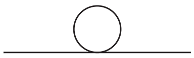
(2.52)

is *connected* (any line is connected to every line). Only connected Feynman diagrams contribute to the non-trivial part of the S matrix.

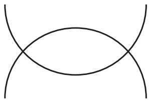
3) The symmetry factor has to be worked out for each graph. However in ϕ^4 a rule can be stated for *connected non-vacuum diagrams* (= graphs with external lines and no disjoint subgraphs). In such a graph, k internal lines are said to form an equivalent set if they all share the same vertices at both ends. If there are more than one such set containing respectively $k_1, k_2, ...$ internal lines, then the symmetry factor is

$$S = \prod_i k_i! \,. \tag{2.53}$$

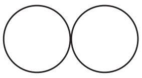
S here is the number of ways in which the lines and the vertices of the graphs can be rearranged without changing its connectivity.
For example we show some diagrams with the corresponding S:



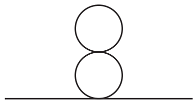
$S = 2$



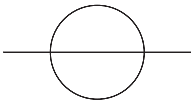
$S = 2$



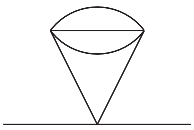
$S = 2 \times 2 \times 2$



$S = 2 \times 2$



$S = 3!$



$S = 3!\,2$

Let us briefly comment on the vacuum graph. At order λ we have

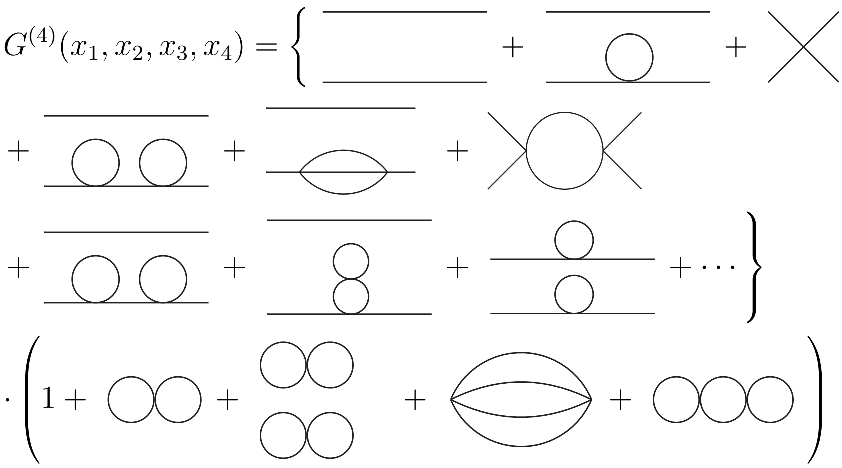
$$= i\frac{\lambda}{4!}\, 3 \int d^4z (D_F(0))^2 \,. \tag{2.54}$$

This involves creation and annihilation of two virtual particle-antiparticle pairs. Leaving out the coefficient we rewrite the vacuum as

$$\begin{aligned} &-i\lambda \int d^4z \, \langle 0|T\phi(0)\phi(0)|0\rangle \langle 0|T\phi(0)\phi(0)|0\rangle \\ &= -i\lambda(2\pi)^4\delta^4(0) \left[\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \right]^2 . \end{aligned} \tag{2.55}$$

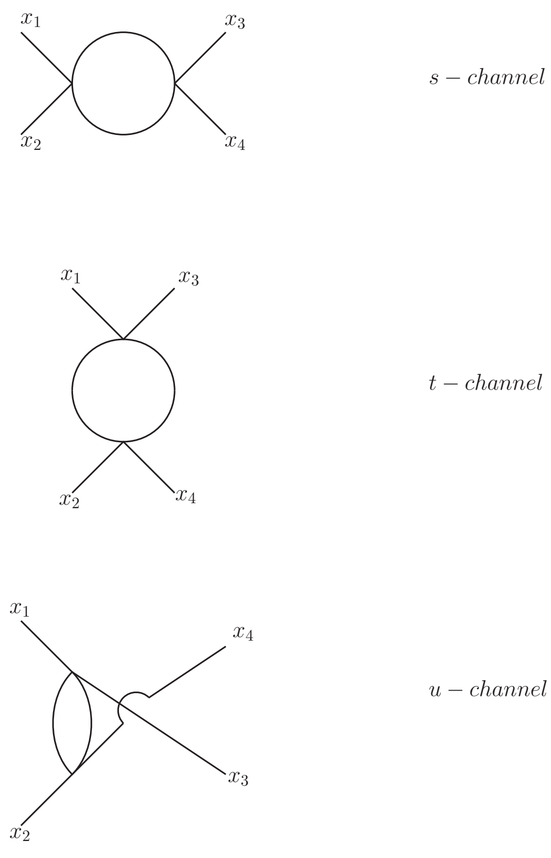
The factor $(2\pi)^4\delta^4(0)$ represents the integral $\int d^4z$ and should be interpreted as the total volume of space-time. Vacuum processes such this one occur *with uniform probability over all space-time* (the vacuum is not empty) and they can accompany any reaction we consider. This factor does not affect physical transition probabilities.

Let us finally present diagrammatically the four-point Green's function $G^{(4)}$ up to order λ^2:



(2.56)

where in the last line of the diagrammatic equation we have the vacuum to vacuum transition that are cancelled by the normalization. At order λ^2 we have three types of connected diagrams:



(2.57)

where the *u-channel* diagram is a non-planar diagram.

2.3 Connected Green's function

The Green's functions of order n are given by the sum of all diagrams with n external legs including the disconnected diagrams but excluding the vacuum diagrams that are

cancelled by the normalization factor. Disconnected graphs do not require separate calculations $\rightarrow$ they are products of lower order connected graphs.
The connected Green's functions are defined as the connected part of $G^{(n)}$: they are obtained by the sum of connected Feynman diagrams. It is typically simpler to work with the connected Green's functions $G^{(n)}_{\text{conn}}(x_1, \dots, x_n)$. Their generating functional is $W[J]$:

$$Z[J] = e^{iW[J]}\,, \quad W[J] = -i \ln Z[J]\,, \tag{2.58}$$

and the connected part of $G^{(n)}$ reads

$$G^{(n)}_{\text{conn}}(x_1, \dots, x_n) = \frac{1}{(i)^n} i \left. \frac{\delta^n W[J]}{\delta J(x_1)\cdots J(x_n)} \right|_{J=0} = \frac{1}{(i)^{n-1}} \left. \frac{\delta^n W[J]}{\delta J(x_1)\cdots J(x_n)} \right|_{J=0} . \tag{2.59}$$

Now we aim to calculate $W[J]$ using the expression found for $Z[J]$ at order λ in (2.41), which is:

$$\begin{aligned} Z[J] = N \Bigg[1 - \int d^4z \left(\frac{i\lambda}{4!}\right) 3(D_F(0))^2 + \frac{i\lambda}{4!} 6 D_F(0) \int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^2 \\ - \frac{i\lambda}{4!} \int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^4 \Bigg] \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\} \\ + \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{2.60}$$

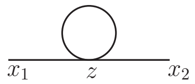
We obtain that

$$\begin{aligned} iW[J] = \ln N - \frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y) - \frac{i\lambda}{4!}\int d^4z\, 3(D_F(0))^2 \\ + \frac{i\lambda}{4!} 6 D_F(0) \int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^2 - \frac{i\lambda}{4!} \int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^4 \\ + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.61}$$

and applying (2.59) to (2.61) we obtain

$$G^{(2)}(x_1, x_2) = G^{(2)}_{\text{conn}}(x_1, x_2) = D_F(x_1 - x_2) - i\frac{\lambda}{2}\int d^4z D_F(x_1 - z) D_F(0) D_F(z - x_2)$$

$=$ 

$- \frac{i\lambda}{2}$ 

, (2.62)

$$G^{(4)}_{\text{conn}}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4z D_F(x_1 - z) D_F(x_2 - z) D_F(x_3 - z) D_F(x_4 - z)$$

$= -i\lambda$  (2.63)

So we see that $G^{(2)}$ is connected as in the free case but now the interaction generates new connected terms. In the free case all connected Green's functions for $n > 2$ were zero. When there is an interaction $G^{(n)}_{\text{conn}}(x_1, \dots, x_n) \neq 0$ for $n > 2$.

Momentum space

Propagators have their simplest form in momentum space. For this reason one typically evaluates the Fourier transform of the Green's functions we have calculated.
The Fourier transform of the Green's functions is defined as:

$$\tilde{G}^{(n)}(p_1, \dots, p_n)(2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) =$$

$$\int d^4x_1 \ldots d^4x_n e^{+ip_1\cdot x_1+\ldots+ip_n\cdot x_n} G^{(n)}(x_1,\, \ldots, x_n)\,, \tag{2.64}$$

where the Dirac delta function occurs because translation invariance implies that $G^{(n)}(x_1,\, \ldots, x_n)$ depends only on the difference of the x_i. Eq. (2.64) corresponds to

$$G^{(n)}(x_1,\, \ldots, x_n) =$$
$$\int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_n}{(2\pi)^4} e^{-ip_1\cdot x_1 - \ldots - p_n\cdot x_n} (2\pi)^4 \delta^4(p_1+p_2+\,\ldots\,+p_n) \tilde{G}^{(n)}(p_1,\, \ldots, p_n)\,. \tag{2.65}$$

All the momenta in (2.64) are treated as outgoing momenta $e^{ip\cdot x}$. Due to the $i\epsilon$ prescription in the propagator, positive particle frequencies propagate into the future $x_n^0 \to \infty$. In actual physical transitions

$$\begin{cases} x_1^0, \ldots, x_k^0 \to -\infty\,, \\ x_{k+1}^0, \ldots, x_n^0 \to +\infty\,, \end{cases} \tag{2.66}$$

and therefore the momenta $p_1, \ldots, p_n$ defined in (2.64) are related to the physical momenta $\tilde{p}_i(\tilde{p}_i^0 > 0)$ of the initial and final particles as follows:

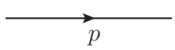
$$\begin{cases} p_i = -\tilde{p}_i\,, & i = 1, \ldots, k \\ p_i = \tilde{p}_i\,, & i = k+1, \ldots, n\,. \end{cases} \tag{2.67}$$

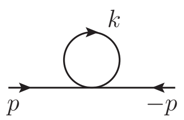
Therefore we have

$$\begin{aligned}
&(2\pi)^4\delta^4(p_1+p_2)\tilde{G}^{(2)}(p_1,p_2) = \int d^4x_1 \int d^4x_2\, e^{ip_1\cdot x_1 + ip_2\cdot x_2} G^{(2)}(x_1,x_2) \\
&= \int d^4x_1 \int d^4x_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2} \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon} \\
&= \int \frac{d^4k}{(2\pi)^4} \delta^4(p_1-k)\delta^4(p_2+k)(2\pi)^4(2\pi)^4 \frac{i}{k^2-m^2+i\epsilon} \\
&= (2\pi)^4\delta^4(p_1+p_2)\frac{i}{p_1^2-m^2+i\epsilon} \\
&\Rightarrow \tilde{G}^{(2)}(p_1,p_2) \equiv G^{(2)}(p_1,-p_1) = \frac{i}{p_1^2-m^2+i\epsilon}\,,
\end{aligned} \tag{2.68}$$

and we can calculate the Fourier transform of (2.63) to get

$$\begin{aligned}
G^{(2)}(p,-p) \;&\equiv\; G^{(2)}(p) \\
&= \frac{i}{p^2-m^2+i\epsilon} - i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{p^2-m^2+i\epsilon} + \mathcal{O}(\lambda^2)
\end{aligned}$$

$=$ 

$+\frac{1}{2}$ 

$+\mathcal{O}(\lambda^2)$ (2.69)

Feynman rules for $\lambda\phi^4$

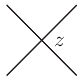
Position space Feynman rules

These are the rules that allow to get the analytic expression for each piece of a Feynman diagram. So the position space Feynman rules read

- for each propagator one has

$$= D_F(x-y)\,, \tag{2.70}$$

- for each vertex one has



$$= (-i\lambda)\int d^4z\,, \tag{2.71}$$

- for each external point one has

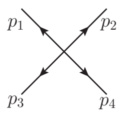
$$= 1 \tag{2.72}$$

- a loop diagram may have some left-over symmetry factor if there are exchanges of internal propagators and vertices that leave the diagram unchanged. In this case we have to divide the value of the diagram by the symmetry factor associated with the exchange of internal propagators and vertices.

The factor $(-i\lambda)$ can be thought as the amplitude for the emission and/or absorption of particles at a vertex. The integral $\int d^4z$ tells us that we have to sum over all points where this process can occur $\rightarrow$ this is the superposition principle of QM $\rightarrow$ when a process can happen in different ways we add the amplitude for each possibility. Moreover in order to calculate each individual amplitude, the Feynman rules tell us to multiply the amplitudes (propagators and vertices) for each of the independent part of the process.

Momentum space Feynman rules

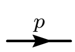
By Fourier transform one can go to momentum space. Then to the propagators one assigns a 4-momentum p, indicating in general the direction of the momentum with an arrow (since in this theory $D_F(x-y) = D_F(y-x)$ the direction of p in the individual propagator line is arbitrary).
The vertex becomes



$$\Leftrightarrow -i\lambda \int d^4z e^{+i(p_1+p_2+p_3+p_4)\cdot z} = -i\lambda\,(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\,. \tag{2.73}$$

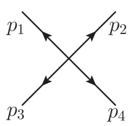
The Feynman rules are:

1) for the propagator (the arrow is arbitrary)



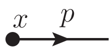
$$= \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{2.74}$$

2) for the vertex



$$= -i\lambda\,, \tag{2.75}$$

3) for the external point



$$= e^{-ipx} \tag{2.76}$$

4) divide by the symmetry factor,

5) impose energy momentum conservation at each vertex,

6) integrate over each undetermined momentum (loop) with a weight

$$\int \frac{d^4k}{(2\pi)^4} \, . \tag{2.77}$$

In order to get the contribution to $\tilde{G}^{(n)}(p_1, \ldots, p_n)$ draw all possible arrangements which are topologically inequivalent after having identified the external legs. The number of ways a given diagram can be drawn is the topological weight of the diagram $1/S$.
In a scalar theory with no derivative coupling the choice of the orientation of an internal line is irrelevant.
If V is the number of vertices and I the number of internal lines, there are $V-1$ conservation rules (coming from conservation of momentum at the vertices and the fact that a global momentum δ should factorize) and so at most there are $I-V+1$ non trivial integration or loops: $L = I - V + 1$.

Notice:
In relativistic QFT virtual transitions conserve both momentum and energy but the squared mass becomes unphysical: $p^2 \neq m^2$ in the propagator inside the loop. For example in the loop diagram in (2.69), recalling the expression as follows

$$-i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{p^2-m^2+i\epsilon} \, , \tag{2.78}$$

we notice that the loop momentum k^2 is not m^2, it is integrated over all k.

2.3.1 loop expansion

One can expand $W[J]$ in terms of the number of loops. This can be done by keep tracking the factor of $\hbar$ in our calculations. For this purpose we start from the functional integral defined in eq. 1.72, and expand it to all orders in perturbation theory. Let us determine the power of $\hbar$ in front of a *connected* Feynman diagram contributing to $W[J]$ for $\lambda\phi^4$. Each propagator has a factor of $\hbar$. Any vertex yields a power of $\hbar^{-1}$. In the same way at the end of all external lines is attached a factor of J which also yields a factor of $\hbar^{-1}$. Calling E the number of external lines (propagator joining a vertex to a source of J), we find that the power

$$\hbar^{I+E-(V+E)+1} \, , \tag{2.79}$$

the last factor of $\hbar$ has been added for our choice of normalization of $W[J]$. Now we use the relation $L = I - V + 1$ to conclude that the power of each diagram is $\hbar^L$; the power of $\hbar$ that we have found counts the number of loops of a diagram.

We have thus shown that the expansion in powers of $\hbar$ is a reordering of perturbation theory according to the number of loops of the Feynman diagrams.

2.4 One particle irreducible Green's functions or connected proper vertex functions

We will be dealing with several type of Feynman diagrams, for instance **connected** and also **amputated** diagrams. The integration associated with a Feynman diagram runs over loop momenta, it is convenient to introduce a reduced diagram which represents the loop integrations by removing all external lines. This is called an amputated diagrams, *i.e.*, without external lines. The external lines can be eliminated by multiplying by the reverse of the corresponding propagators.

The connected proper vertex function $\Gamma^{(n)}(x_1, \dots, x_n)$, which are also called one-particle-irreducible (1PI) Green's functions, are given by the amputated Feynman diagrams that are one-particle-irreducible, *i.e.*, they remain connected after that an arbitrary internal line is cut. In lowest order (*i.e.*, with no loop considered) the connected proper vertex functions coincide (when appropriately normalized) with the vertices of the original Lagrangian.

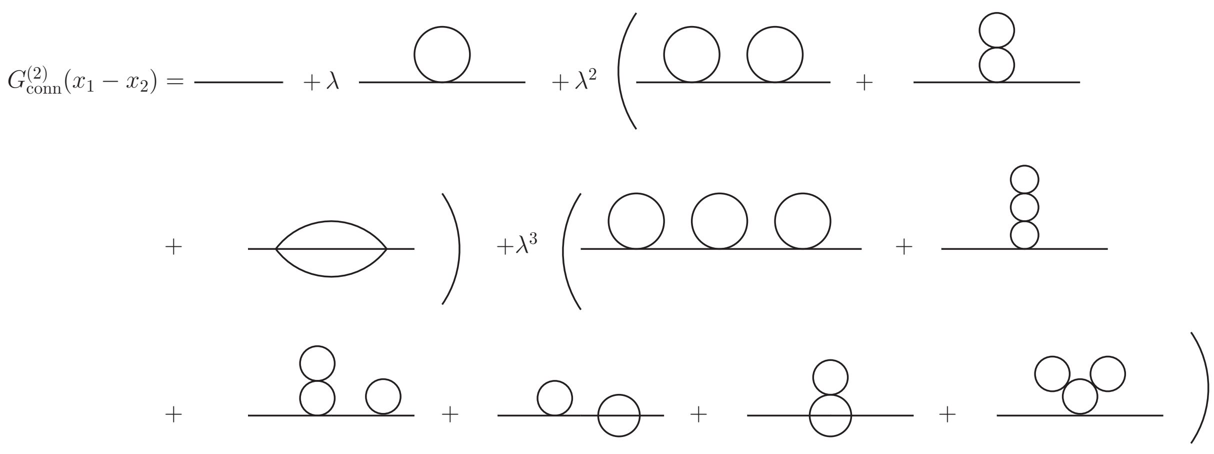
The connected proper vertex functions are the building blocks of perturbation theory. We can define the generating functional Γ of the $\Gamma^{(n)}(x_1, \dots, x_n)$:

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots \int d^4x_n \Gamma^{(n)}(x_1, \dots, x_n)\phi(x_1)\dots\phi(x_n)\,, \tag{2.80}$$

with

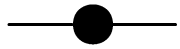
$$\Gamma^{(n)}(x_1, \dots, x_n) = \left.\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)}\right|_{\phi=0}\,. \tag{2.81}$$

$\Gamma[\phi]$ is called *effective action*. We want to see how $\Gamma[\phi]$ is related to $W[J]$.
Before doing this let us comment about the role of 1PI Green's functions with a concrete example. We consider the two-point connected Green's function up to order λ^4. Schematically (without indicating the numerical factors or the i) we have



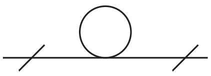
(2.82)

These are all connected diagrams and we would like to derive a method to sum them. The sum of all these diagrams is called complete or dressed propagator and is denoted as follows:

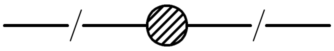
$$G^{(2),\text{full}}_{\text{conn}}(x_1 - x_2) =$$  . (2.83)

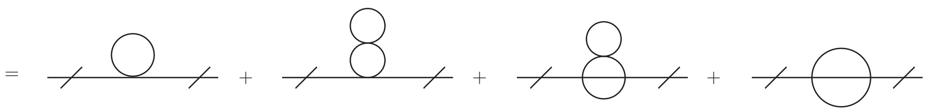
We have seen that the effect of order λ and higher orders is to change the physical mass away from the "bare mass" m and hence to give rise to a self-energy. The graphs above all contribute to the self-energy. The reason for selecting 1PI diagrams is that one particle reducible diagrams can be decomposed into 1PI diagrams without further loop integration and if we know how to take care of the 1PI divergencies then also those of the reducible diagrams will be handled $\rightarrow$ to make the theory finite we need to make finite only the 1PI diagrams.

We define amputated diagrams by multiplying the external legs by inverse propagators, diagrammatically

 .

Then of the three graphs of order λ^2, the first is a product of lower order graphs but the other two are not. This is because the first graph contains a propagator $\rightarrow$ it is called a 1P reducible graph. This is not the case for the other two graphs which are 1PI. Based on this classification we can define *a proper self-energy part* as the sum of the amputated 1PI graphs:

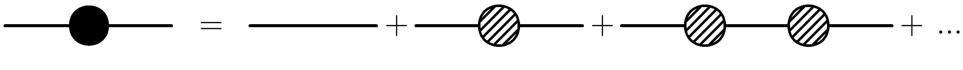
 $= -i\Sigma(p) =$

 . (2.84)

The complete propagator in (2.82) may therefore be written in terms of the bare propagator $G_0(p) = i/(p^2 - m^2)$ and the proper self energy function $\sum(p)$ as follows:

$$\begin{aligned}
G^{(2)}_{\text{conn}}(p) &= G_0(p) + G_0(p)\left(-i\Sigma(p)\right)G_0(p) + G_0(p)\left(-i\Sigma(p)\right)G_0(p)\left(-i\Sigma(p)\right)G_0(p) + ... \\
&= G_0(p)\left(1 - i\Sigma(p)G_0(p) - i\Sigma(p)G_0(p)\left(-i\Sigma(p)\right)G_0(p) + ...\right) \\
&= G_0(p)\left(1 + i\Sigma(p)G_0(p)\right)^{-1} = \left[G_0^{-1}(p) + i\Sigma(p)\right]^{-1} \\
&= \frac{1}{\frac{p^2-m^2}{i} - \frac{\Sigma(p)}{i}} = \frac{i}{p^2 - m^2 - \Sigma(p)}\,, \qquad (2.85)
\end{aligned}$$

or in diagrams:

 (2.86)

Defining the physical mass by the pole of the complete propagator

$$G^{(2)}_{\text{conn}}(p) = \frac{i}{p^2 - m^2_{\text{phys}}}\,, \qquad (2.87)$$

we obtain

$$m^2_{\text{phys}} = m^2 + \Sigma(p)\,, \qquad (2.88)$$

which justifies the name of "self-energy" that we have given to Σ. It represents the change in mass from the bare to the physical value calculated to all orders in perturbative theory. From (2.85) we have

$$\left(G^{(2)}_{\rm conn}(p)\right)^{-1} = G_0(p)^{-1} + i\Sigma(p)\,. \tag{2.89}$$

So the inverse of two-point function contains only 1PI graphs (apart from the inverse bare propagator). This is an example of proper vertex function that may be generalized. The two-point vertex function $\Gamma^{(2)}(p)$ is defined by

$$\begin{aligned} &G^{(2)}_{\rm conn}(p)\Gamma^{(2)}(p) = i \\ &\Rightarrow \Gamma^{(2)}(p) = p^2 - m^2 - \Sigma(p)\,, \end{aligned} \tag{2.90}$$

where in the last step we use the result of (2.85).

Now we want to obtain the generating functional for the $\Gamma^{(n)}$, and we will see the double correspondence: $\Gamma[\phi] \Leftrightarrow W[J]$ and $\phi \Leftrightarrow J$. To this aim we consider the vacuum expectation value of the field $\phi(x)$ in presence of an external source $J(x)$

$$\phi_{\rm cl}(x) = \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \tag{2.91}$$

We can derive eq. (2.91) as follows:

$$\begin{aligned} \phi_{\rm cl}(x) &= \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{1}{\hbar}\frac{\hbar}{i}\frac{\delta \ln Z[J]}{\delta J(x)} = \frac{1}{i}\frac{1}{Z}\frac{\delta Z[J]}{\delta J(x)} \\ &= \frac{1}{i}\frac{\int \mathcal{D}\phi\, i\phi(x) exp\left\{iS[\phi] + i\int d^4x J\phi\right\}}{\int \mathcal{D}\phi\, exp\left\{iS[\phi] + i\int d^4x J\phi\right\}} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \end{aligned} \tag{2.92}$$

The vacuum expectation value $\langle\phi\rangle = \langle 0|\phi|0\rangle/\langle 0|0\rangle$ is the limit of $\phi_{\rm cl}$ as $J(x) \to 0$. Here $\phi_{\rm cl}$ is determined by the external source J (it is a functional of the external source J). $\phi_{\rm cl}$ is an ordinary function and we can think of it as a classical field.

We now ask a question: what source J will produce a given function $\phi_{\rm cl}$? To answer we reformulate the problem so that it becomes convenient to use $\phi_{\rm cl}$ as independent variable in place of J. We define the "effective action" $\Gamma[\phi_{\rm cl}]$ by making a Legendre transformation

$$\Gamma[\phi_{\rm cl}] = W[J] - \hbar\int d^4x J(x)\phi_{\rm cl}(x)\,, \tag{2.93}$$

where J has to be eliminated in terms of $\phi_{\rm cl}$ by solving (2.91). $\Gamma[\phi_{\rm cl}]$ *is called effective action because it is a functional of the classical field* $\phi_{\rm cl}$ *and thus similar to* $S[\phi]$. The classical field $\phi_{\rm cl}$ minimizes the combination

$$\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\,, \tag{2.94}$$

or in other words is a solution of the "classical field" equations:

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = -\hbar J(x)\,. \tag{2.95}$$

(Notice that this equation is the reverse of eq. (2.91).) By differentiating eq. (2.93) with respect to $J(x)$ we find

$$\frac{\delta W[J]}{\delta J(x)} = \int d^4y \underbrace{\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(y)}}_{-\hbar J(y)} \frac{\delta\phi_{\rm cl}(y)}{\delta J(x)} + \hbar\,\phi_{\rm cl}(x) + \hbar\int d^4y \frac{\delta\phi_{\rm cl}(y)}{\delta J(x)} J(y)\,, \tag{2.96}$$

where we used (2.95), and then we obtain

$$\phi_{\rm cl}(x) = \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \tag{2.97}$$

If $W[J]$ is known we can use (2.91) to determine $J(x)$ in terms of $\phi_{\rm cl}$ so that (2.93) can be written as a functional of $\phi_{\rm cl}$ which determines $\Gamma[\phi_{\rm cl}]$. The value in (2.91) when the external source is turned off ($J(x) = 0$) is the vacuum expectation value of $\phi(x)$ which by now we assume to be zero

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = 0\,. \tag{2.98}$$

The case of nonzero vacuum expectation value reflects spontaneous symmetry breaking and will be discussed later. Eq. (2.95) gives $J(x)$ in terms of $\phi_{\rm cl}(x)$ and in this sense is the inverse of (2.91). If we combine (2.95) and (2.98) we obtain

$$\left.\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}\right|_{\phi_{\rm cl}=0} = 0\,, \tag{2.99}$$

because when $J = 0$ it follows $\phi_{\rm cl} = 0$ and viceversa.
We can see then why $\Gamma^{(n)}$ is the n-point 1PI Green's function, *i.e*, the sum of all amputated 1PI Feynman diagram with n external legs. Let us consider

$$N\int\mathcal{D}\phi\,\exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\right]\right\}\,, \tag{2.100}$$

which is the generating functional of a new set of Green's functions corresponding to the new action $\Gamma[\phi]$. For $\hbar \to 0$ the integrand is dominated by the saddle point which occurs at $\phi = \phi_{\rm cl}$ due to

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = -\hbar J(x)\,, \tag{2.101}$$

which is precisely the saddle point condition. Thus

$$\begin{aligned} & N\int\mathcal{D}\phi\,\exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\right]\right\} \\ & \xrightarrow{\ \hbar\to 0\ } \quad N'\exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi_{\rm cl}] + \hbar\int d^4x J(x)\phi_{\rm cl}(x)\right]\right\} \end{aligned} \tag{2.102}$$

and by (2.93) this is equal to

$$N'\exp(\frac{i}{\hbar}W[J])\,. \tag{2.103}$$

Therefore, the generating functional corresponding to $\Gamma[\phi]$ in the limit $\hbar \to 0$ is equal to $N'\exp(\frac{i}{\hbar}W[J])$. On the other hand, in the limit $\hbar \to 0$ only the tree level graphs of the new theory, whose action is $\Gamma[\phi]$, survive. Hence a vertex in one of these tree graphs is some $\Gamma^{(n)}$ by definition, whose form is given in (2.80). Moreover, recall that $W[J]$ is by definition the sum of all connected Green's function of the original theory and a connected Green's function can be dissected into 1PI components. So $\Gamma^{(n)}$ is a 1PI Green's function of the original theory. This will be explained more later.

Notice:
In the free case we have ($\hbar = 1$)

$$W[J] = -i \ln Z = +\frac{i}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \,, \tag{2.104}$$

and we obtain

$$\phi_{\rm cl}(x) = \frac{\delta W[J]}{\delta J(x)} = i \int d^4x' D_F(x-x') J(x') \,, \tag{2.105}$$

so we see immediately that

$$\begin{aligned}(\partial^2 + m^2)\phi_{\rm cl}(x) &= i \int d^4x' \underbrace{(\partial_x^2 + m^2) D_F(x-x')}_{-i\delta^4(x-x')} J(x') = \\ &= J(x) \,,\end{aligned} \tag{2.106}$$

which is identical to the classical field equation in presence of a source J and thus tells us why it is appropriate to refer to $\phi_{\rm cl}$ as the classical field.
Moreover we can calculate $\Gamma[\phi_{\rm cl}]$ explicitly as we know $W[J]$ and because (2.106) allows us to eliminate J in favour of $\phi_{\rm cl}$. Starting from

$$\Gamma[\phi_{\rm cl}] = \frac{i}{2} \int d^4x \int d^4x' J(x) D_F(x-x') J(x') - \int d^4x J(x) \phi_{\rm cl}(x) \tag{2.107}$$

and substituting $J(x)$ from (2.106) we obtain

$$\begin{aligned}\Gamma[\phi_{\rm cl}] &= \frac{i}{2} \int d^4x \int d^4x' \Big((\partial_x^2 + m^2)\phi_{\rm cl}(x)\Big) D_F(x-x') \Big((\partial_{x'}^2 + m^2)\phi_{\rm cl}(x')\Big) \\ &- \int d^4x \phi_{\rm cl}(x)(\partial_x^2 + m^2)\phi_{\rm cl}(x) \\ &= \frac{1}{2} \int d^4x \phi_{\rm cl}(x)(\partial_x^2 + m^2)\phi_{\rm cl}(x) - \int d^4x \phi_{\rm cl}(x)(\partial_x^2 + m^2)\phi_{\rm cl}(x) \\ &= -\frac{1}{2} \int d^4x \phi_{\rm cl}(x)(\partial_x^2 + m^2)\phi_{\rm cl}(x) = \frac{1}{2} \int d^4x (\partial_\mu \phi_{\rm cl} \partial^\mu \phi_{\rm cl} - m^2 \phi_{\rm cl}^2) \,,\end{aligned} \tag{2.108}$$

which is exactly the classical free-field theory action and justifies referring to $\Gamma[\phi_{\rm cl}]$ as the effective action. Note that we have used integration by part and the relation $(\partial_x^2 + m^2) D_F(x-x') = -i\delta(x-x')$ to derive eq. (2.108). In the case of an interacting theory we will not be able to calculate $\phi_{\rm cl}$ and $\Gamma[\phi_{\rm cl}]$ exactly and there will be quantum correction to (2.106) and (2.108).
In the free theory we see, using eq. (2.108) and (2.81), that the only $\Gamma^{(n)} \neq 0$ is the $\Gamma^{(2)}$:

$$\Gamma^{(2)}(x,x') = -(\partial_{x'}^2 + m^2)\delta^4(x-x') \,, \tag{2.109}$$

with Fourier transform

$$(2\pi)^4 \delta^4(p_1+p_2) \tilde{\Gamma}^{(2)}(p_1,p_2) = \int d^4x_1 \int d^4x_2 e^{ip_1\cdot x_1 + ip_2 \cdot x_2} \Gamma(x_1,x_2)$$

$$= \int d^4x_1 \int d^4x_2 e^{ip_1\cdot x_1 + ip_2\cdot x_2}(-1)(\partial^2_{x_1} + m^2)\delta^4(x_1 - x_2)$$
$$= (-1)(-p_1^2 + m^2)(2\pi)^4\delta^4(p_1 + p_2)$$
$$\Rightarrow \tilde{\Gamma}^{(2)}(p_1, p_2) = (p_1^2 - m^2)\,, \tag{2.110}$$

confirming that

$$G^{(2)}_{\rm conn}(p)\Gamma^{(2)}(p) = i\,. \tag{2.111}$$

Notice that (2.93) is completely analogous to the thermodynamic equation $U = F + TS$ which gives the internal energy U as a function of the entropy S in terms of the free energy F regarded as a function of the temperature T.

Classical field and effective action at order λ

Using the expression for $W[J]$:

$$W[J] = +i\ln N + \frac{i}{2}\int d^4x \int d^4y J(x) D_F(x-y)J(y) - \frac{\lambda}{4!}\int d^4z\, 3(D_F(0))^2$$
$$+\frac{\lambda}{4!}6D_F(0)\int d^4z\left(\int d^4z' D_F(z-z')J(z')\right)^2 - \frac{\lambda}{4!}\int d^4z\left(\int d^4z' D_F(z-z')J(z')\right)^4\,, \tag{2.112}$$

and using the definition of classical field $(\delta W[J])/(\delta J)$ we get

$$\phi_{\rm cl}(x) = +i\int d^4y D_F(x-y)J(y) + \frac{1}{2}\lambda\int d^4z\, d^4z'\, D_F(x-z)D_F(z-z)D_F(z-z')J(z')$$
$$-\frac{\lambda}{6}\int d^4z d^4z_1' d^4z_2' d^4z_3' D_F(x-z)D_F(z_1'-z)D_F(z_2'-z)D_F(z_3'-z)J(z_1')J(z_2')J(z_3')$$
$$+\mathcal{O}(\lambda^2)\,, \tag{2.113}$$

and we obtain for the effective action

$$\Gamma[\phi_{\rm cl}] = W[J] - \int d^4 J(x)\phi_{\rm cl}(x)$$
$$= +i\ln N - \frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2)J(y_1)J(y_2) - \frac{1}{8}\lambda\int d^4x[D_F(0)]^2$$
$$-\frac{\lambda}{4}\int d^4x d^4y_1\, d^4y_2\, D_F(y_1-x)D_F(x-x)D_F(x-y_2)J(y_1)J(y_2)$$
$$+\frac{1}{8}\lambda\int d^4x d^4y_1\, d^4y_2\, d^4y_3\, d^4y_4\, D_F(y_1-x)D_F(y_2-x)D_F(y_3-x)D_F(y_4-x)J(y_1)J(y_2)J(y_3)J(y_4)$$
$$+\mathcal{O}(\lambda^2)\,. \tag{2.114}$$

Eq. (2.113) can be solved for $J(x)$ perturbatively and by performing integration by parts using

$$(\partial_x^2 + m^2)D_F(x-y) = -i\delta^4(x-y)\,. \tag{2.115}$$

To calculate $J(x)$ we start off by acting the operator $(\partial_x^2 + m^2)$ on eq. (2.113)

$$(\partial_x^2 + m^2)\phi_{\rm cl}(x) = +i\int d^4y(\partial_x^2 + m^2)D_F(x-y)J(y)$$
$$+\frac{1}{2}\lambda\int d^4z\, d^4z'\,(\partial_x^2 + m^2)D_F(x-z)D_F(z-z)D_F(z-z')J(z')$$

$$-\frac{\lambda}{6}\int d^4z d^4z_1' d^4z_2' d^4z_3' (\partial_x^2+m^2) D_F(x-z) D_F(z_1'-z) D_F(z_2'-z) D_F(z_3'-z) J(z_1') J(z_2') J(z_3')$$
$$+\mathcal{O}(\lambda^2)\,, \tag{2.116}$$

which reads

$$(\partial_x^2+m^2)\phi_{\rm cl}(x) = J(x) + \frac{-i\lambda}{2}\int d^4z'\, D_F(0) D_F(x-z') J(z')$$
$$-\frac{-i\lambda}{6}\int d^4z_1' d^4z_2' d^4z_3' D_F(z_1'-x) D_F(z_2'-x) D_F(z_3'-x) J(z_1') J(z_2') J(z_3')$$
$$+\mathcal{O}(\lambda^2)\,. \tag{2.117}$$

This relation can be organized as

$$J(x) = (\partial_x^2+m^2)\phi_{\rm cl}(x) - \frac{-i\lambda}{2}\int d^4z'\, D_F(0) D_F(x-z') J(z')$$
$$+\frac{-i\lambda}{6}\int d^4z_1' d^4z_2' d^4z_3' D_F(z_1'-x) D_F(z_2'-x) D_F(z_3'-x) J(z_1') J(z_2') J(z_3')$$
$$-\mathcal{O}(\lambda^2)\,. \tag{2.118}$$

Now we arrive to the point that it is straightforward to calculate $J(x)$ perturbatively (substituting, for instance, the zero order identification of $J(x)$ in terms of $\phi_{\rm cl}(x)$ when working at order λ). At zero order we have

$$J(x) = (\partial^2+m^2)\phi_{\rm cl}(x) + \mathcal{O}(\lambda)\,, \tag{2.119}$$

and at first order in λ we obtain

$$J(x) = (\partial^2+m^2)\phi_{\rm cl}(x) + \frac{\lambda}{2} D_F(0)\phi_{\rm cl}(x) + \frac{\lambda}{6}[\phi_{\rm cl}(x)]^3 + \mathcal{O}(\lambda^2)\,. \tag{2.120}$$

From (2.120) one can verify that $\phi_{\rm cl}$ satisfies an equation similar to

$$(\partial_x^2+m^2)\phi(x) = -\frac{\lambda}{6}\phi^3(x)\,, \tag{2.121}$$

with the external source added and quantum corrections:

$$(\partial_x^2+m^2)\phi_{\rm cl}(x) = J(x) - \frac{\lambda}{2} D_F(0)\phi_{\rm cl} - \frac{\lambda}{6}\phi_{\rm cl}^3(x)\,. \tag{2.122}$$

If we restore $\hbar$ we would see that the second term in the last equation is proportional to $\hbar$. Substituting (2.120) into (2.114) gives $\Gamma[\phi_{\rm cl}]$ explicitly as a functional of $\phi_{\rm cl}(x)$. Using the following relations:

$$-\frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2) J(y_1) J(y_2)$$
$$= -\frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2)\left((\partial_{y_1}^2+m^2)\phi_{\rm cl}(y_1) + \frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_1) + \frac{\lambda}{6}[\phi_{\rm cl}(y_1)]^3\right)$$
$$\times\left((\partial_{y_2}^2+m^2)\phi_{\rm cl}(y_2) + \frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_2) + \frac{\lambda}{6}[\phi_{\rm cl}(y_2)]^3\right) + \mathcal{O}(\lambda^2)$$
$$= -\frac{1}{2}\int d^4y_1\, \phi_{\rm cl}(y_1)(\partial_{y_1}^2+m^2)\phi_{\rm cl}(y_1)$$

$$-\frac{2}{2}\int d^4y_1\,\phi_{\rm cl}(y_1)\left(+\frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_1)+\frac{\lambda}{6}[\phi_{\rm cl}(y_1)]^3\right)+\mathcal{O}(\lambda^2)\,, \tag{2.123}$$

$$\begin{aligned}&-\frac{\lambda}{4}\int d^4xd^4y_1\,d^4y_2\,D_F(y_1-x)D_F(x-x)D_F(x-y_2)J(y_1)J(y_2)\\&=+\frac{\lambda}{4}\int d^4x\,D_F(0)\,(\phi_{\rm cl}(y_1))^2+\mathcal{O}(\lambda^2)\,,\end{aligned} \tag{2.124}$$

$$\begin{aligned}&+\frac{1}{8}\lambda\int d^4xd^4y_1\,d^4y_2\,d^4y_3\,d^4y_4\,D_F(y_1-x)D_F(y_2-x)D_F(y_3-x)D_F(y_4-x)J(y_1)J(y_2)J(y_3)J(y_4)\\&=+\frac{\lambda}{8}\int d^4x\,D_F(0)\,(\phi_{\rm cl}(y_1))^4+\mathcal{O}(\lambda^2)\,,\end{aligned} \tag{2.125}$$

one can show that

$$\begin{aligned}\Gamma[\phi_{\rm cl}]&=+i\ln N-\frac{\lambda}{8}\int d^4z(D_F(0))^2-\frac{1}{2}\int d^4x\,\phi_{\rm cl}(x)(\partial_x^2+m^2)\phi_{\rm cl}(x)\\&-\frac{\lambda}{4}D_F(0)\int d^4x(\phi_{\rm cl}(x))^2-\frac{\lambda}{4!}\int d^4x(\phi_{\rm cl}(x))^4+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{2.126}$$

From this equation and from

$$\Gamma^{(n)}(x_1,\,\dots,x_n)=\left.\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)}\right|_{\phi=0}\,, \tag{2.127}$$

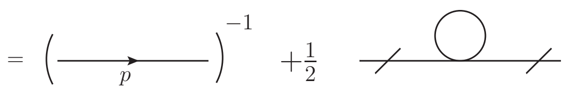
we obtain that

$$\Gamma^{(2)}(x_1,x_2)\;=\;-\left(\partial_{x_1}^2+m^2+\frac{\lambda}{2}D_F(0)\right)\delta(x_1-x_2)\,, \tag{2.128}$$

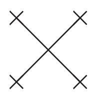
$$\Gamma^{(4)}(x_1,x_2,x_3,x_4)\;=\;-\lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)\,, \tag{2.129}$$

we obtain in momentum space

$$i\Gamma^{(2)}(p,-p)=i(p^2-m^2)-i\frac{\lambda}{2}D_F(0)+\mathcal{O}(\lambda^2)$$



(2.130)

$i\Gamma^{(4)}(p_1,p_2,p_3,p_4)=-i\lambda$  . (2.131)

Relation between $G_{\rm conn}^{(n)}$ and $\Gamma_{\rm conn}^{(n)}$

We recall that (we set $\hbar=1$)

$$\Gamma[\phi_{\rm cl}]=W[J]-\int d^4xJ(x)\phi_{\rm cl}(x)\,, \tag{2.132}$$

and we have got

$$\frac{\delta W[J]}{\delta J(x)} = \phi_{\text{cl}}(x)\,, \quad \frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)} = -J(x)\,. \tag{2.133}$$

Therefore we get for

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta \phi_{\text{cl}}(x)}{\delta J(y)}\,, \quad G^{(2)}_{\text{conn}}(x,y) = \frac{1}{i}\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0}\,, \tag{2.134}$$

$$\frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)\phi_{\text{cl}}(y)} = -\frac{\delta J(x)}{\delta \phi_{\text{cl}}(y)}\,, \quad \Gamma^{(2)}(x,y) = \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)\phi_{\text{cl}}(y)}\bigg|_{J=0}\,. \tag{2.135}$$

These are clearly one the inverse of the other

$$\int d^4z(-i)\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\phi_{\text{cl}}(y)} = i\int d^4z\frac{\delta \phi_{\text{cl}}(x)}{\delta J(z)}\frac{\delta J(x)}{\delta \phi_{\text{cl}}(y)}$$
$$= \frac{\delta \phi_{\text{cl}}(x)}{\delta \phi_{\text{cl}}(y)} = i\delta^4(x-y)\,, \tag{2.136}$$

which is then equivalent to (by using the relations in eqs. (2.135))

$$\int d^4z G^{(2)}_{\text{conn}}(x,z)\Gamma^{(2)}(z,y) = i\delta^4(x-y)\,, \tag{2.137}$$

$$\Rightarrow G^{(2)}_{\text{conn}}(x,y) = i\left(\Gamma^{(2)}\right)^{-1}(x,y)\,. \tag{2.138}$$

We can obtain many relations of this type simply by differentiating (2.136) and using

$$\frac{\delta}{\delta J(x)} = \int d^4y\frac{\delta \phi_{\text{cl}}(y)}{\delta J(x)}\frac{\delta}{\delta \phi_{\text{cl}}(y)} = \int d^4y\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\frac{\delta}{\delta \phi_{\text{cl}}(y)}\,. \tag{2.139}$$

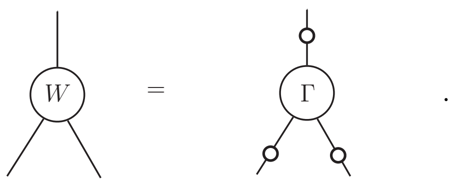
For example let us differentiate eq. (2.136) with respect to $\delta/\delta J(x_1)$:

$$\int d^4z\left(-\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x)\delta J(z)}\right)\frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\phi_{\text{cl}}(y)}$$
$$+\int d^4z\int d^4x'\left(-\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\right)\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x')}\frac{\delta^3 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(z)\delta \phi_{\text{cl}}(y)\delta \phi_{\text{cl}}(x')} = 0\,. \tag{2.140}$$

If we now multiply by $(\delta^2 W[J])(\delta J(y)\delta J(x_3))$ and perform the integral over y and use (2.136) we will have (redefining labels)

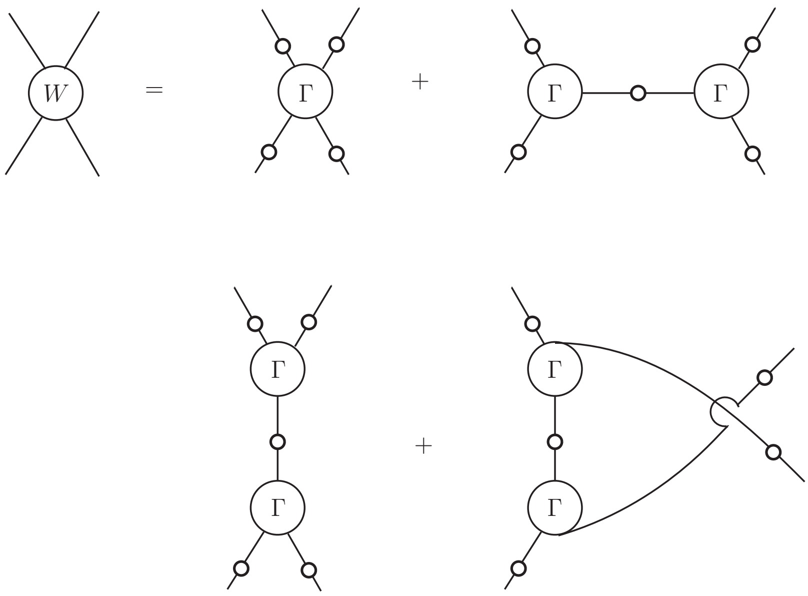
$$\frac{\delta^3 W[J]}{\delta J(y_1)\delta J(y_2)\delta J(y_3)}$$
$$= \int d^4x_1\int d^4x_2\int d^4x_3\frac{\delta^2 W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^2 W[J]}{\delta J(y_3)\delta J(x_3)}$$
$$\times\frac{\delta^3 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x_1)\delta \phi_{\text{cl}}(x_2)\delta \phi_{\text{cl}}(x_3)}\,. \tag{2.141}$$

Eq. (2.141) can be represented as ($\bigcirc$ exact propagators in the external lines)



(2.142)

Differentiating again eq. (2.141) we obtain

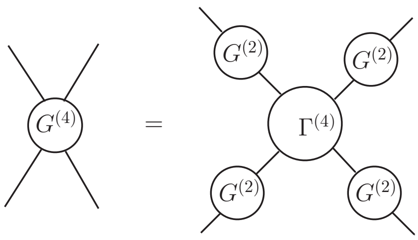


(2.143)

When $J = 0$ several contributions go to zero since, for example, we do not have in $\lambda\phi^4$ theory *1PI diagrams with an odd number of external lines*:

$$\frac{\delta^3\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)\delta\phi_{\text{cl}}(y)\delta\phi_{\text{cl}}(z)} = 0\,. \tag{2.144}$$

Then we have:

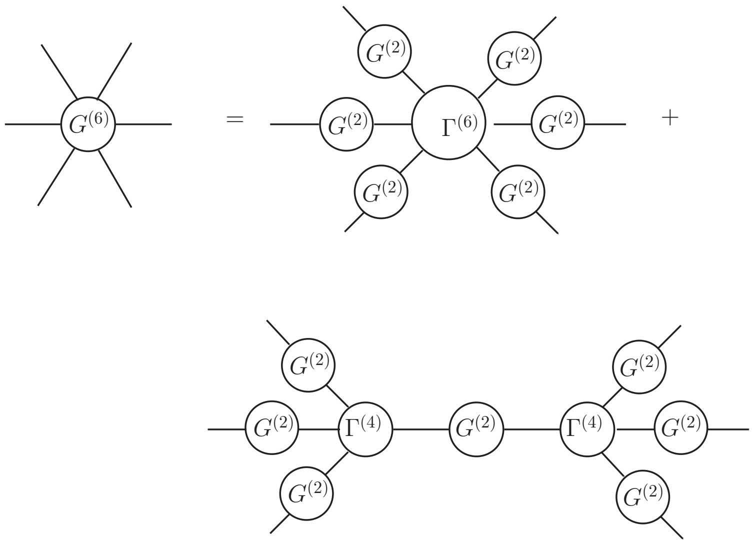


(2.145)

which one writes analytically:

$$\begin{aligned}
&G^{(4)}_{\text{conn}}(y_1, y_2, y_3, y_4) \\
&= \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 G^{(2)}_{\text{conn}}(x_1 - y_1)G^{(2)}_{\text{conn}}(x_2 - y_2)G^{(2)}_{\text{conn}}(x_3 - y_3)G^{(2)}_{\text{conn}}(x_4 - y_4) \\
&\times i\Gamma^{(4)}(x_1, x_2, x_3, x_4)\,,
\end{aligned} \tag{2.146}$$

and diagrammatically:



(2.147)

These formulas show *the perturbative meaning of* $\Gamma^{(n)}(x_1, \ldots, x_n)$*: 1PI diagrams with amputated external legs.* The only exception is $\Gamma^{(2)}$.

3 Effective potential and spontaneous symmetry breaking

3.1 Introduction

The notion of the effective potential (the potential part of the effective action) is particularly useful for theories with a spontaneous symmetry breaking (SSB).
What is a SSB? SSB means that the vacuum of the theory, i.e. the field configuration with the minimal energy, does not posses the symmetry of the Lagrangian.
But how can we determine the vacuum of the theory? In a classical field theory we can use the potential part of $\mathcal{L}$ to decide what is the field configuration ϕ_0 of minimal energy via the condition

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\phi_0} = 0\,. \tag{3.1}$$

In QFT we have quantum corrections and we have to calculate such corrections before finding the vacuum. In this case it is useful to use the effective potential. In some cases quantum corrections to the classical potential will spontaneously break a symmetry. Quantum corrections may indeed produce a non zero vacuum expectation value (v.e.v.) of the field where the classical vacuum is zero (as in scalar massless electrodynamics).

History. Around 1960 Nambu and Goldstone realized the importance of the concept of SSB in condensed matter and Nambu speculated on its applications to particle physics. In 1964 Brout, Englert, and Higgs realized that the consequences of SSB in gauge theories are very different with respect to non-gauge theories. Later Glashow, Weinberg and Salam applied Brout–Englert–Higgs' ideas to a SU(2)×U(1) gauge theory. This led to the unification of weak and electromagnetic interactions in the Standard Model of particle physics, which is remarkably reproduced by the experimental data.
There are different types of SSB:

- SSB of a discrete symmetry ,
- SSB of a continuous symmetry → Goldstone bosons, (for example SSB of chiral symmetry in QCD, which leads to pion physics)
- SSB of a gauge symmetry → gauge field acquire a mass, (examples are supperconductivity and the physics of Higgs particle in the SM).

A typical and intuitive example of SSB comes from condensed matter physics: ferromagnetic materials and ferromagnetic/paramagnetic transitions. Let us recall ferromagnetism: certain materials (iron, nickel, cobalt) can be magnetized at room temperature. Microscopically this means that the electrons in an incomplete inner shell have their spins aligned in the same direction. Since there is a magnetic moment associated with the spins, this means that the single magnetic moments add producing a magnet with magnetization $\langle M\rangle \neq 0$. This happens in domains of $\approx 10^{-2}$ mm and the total magnetization $\langle M\rangle$ of the whole sample can be zero (see figure 3.1). So in a ferromagnetic material in absence of external magnetic field ($B = 0$) a spontaneous magnetization exists inside each

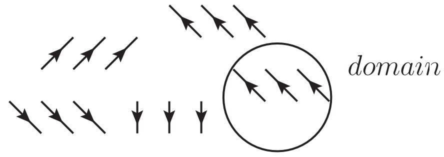


Figure 3.1: Domains in a ferromagnet.

domain. The vacuum state breaks the symmetry of the Lagrangian, whose corresponding Hamiltonian in this case has the form

$$H = -\sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j \quad \text{(Heisenberg Hamiltonian)}\,, \tag{3.2}$$

and displays a rotational symmetry. However the ground state is not invariant under this symmetry (it has a preferred direction), rather, under this symmetry it changes into one of the other degenerate ground states. (In a ferromagnet the ground states are the states in which all the spins inside a domain are aligned. But, the direction of $\langle M\rangle$ is random and all degenerate ground states may be realized starting from one and applying a rotation.)

The state of a domain is no longer invariant under all rotations but only under rotations around axes parallel to the magnetization direction. Therefore the symmetry group is only a subgroup of the original one. In other words the original symmetry is broken spontaneously without applying any external magnetic field.

When a ferromagnet is heated above a certain temperature T_c ($\sim 10^3$ K, Curie temperature), its magnetization disappears and the material becomes paramagnetic (thermal agitation tends to destroy the initial configuration).
$\Rightarrow$ the magnetization of the domain goes to zero (see figure 3.2).
The magnetization behaves as the order parameter of the transition (second order transition). The $\langle M\rangle \neq 0$ at $B = 0$ of this example is the equivalent of a v.e.v. $\langle \phi\rangle \neq 0$ at

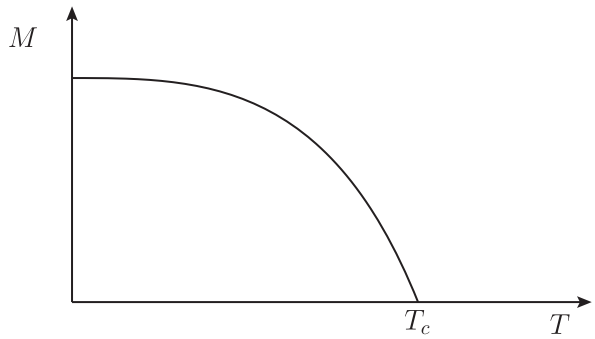


Figure 3.2: Magnetization versus temperature and Curie temperature.

$J = 0$, and $\langle\phi\rangle$ is the order parameter of the phase transition which is in the Lagrangian. Examples are the Higgs condensate $\langle\phi\rangle = 0$ at $T > T_c$ in the electroweak phase transition of the early universe, or $\langle\bar{q}q\rangle = 0$ in QCD at the deconfinement transition (chiral phase transition)).

In summary, here we are dealing with some system that possesses a symmetry (for example for ferromagnetic systems the rotational symmetry: $H = -\sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$) but the ground state is not invariant under this symmetry, rather, under this symmetry it changes into one of the other degenerate ground states (for example in a ferromagnet the ground state is one in which all the spins, within domain, are aligned. This is clearly not rotationally invariant. The direction of the spontaneous magnetization is random and the degenerate ground states may be reached from a given one by rotation).

Notice:
for a relativistic theory the vacuum is Poincaré invariant and this implies that only scalar combinations of field operators may have a v.e.v.

So in general the v.e.v. may break the internal symmetry of $\mathcal{L} \to$ the vacuum has less symmetry than the theory allows.

3.2 $\lambda\phi^4$ with classical fields

Let us study the state of minimal energy (the vacuum state) given the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V[\phi(x)]\,, \quad V[\phi(x)] = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\,. \tag{3.3}$$

The minimum energy classical configuration is a constant field $\phi = v$ such that

$$\left.\frac{\partial V[\phi]}{\partial\phi}\right|_{\phi=v} = 0\,, \tag{3.4}$$

where $\phi = v$ corresponds in QFT to the existence of a non vanishing v.e.v., namely $\langle 0|\phi(x)|0\rangle = v$. The condition (3.4) for a constant field is equivalent to be a solution of the equations of motion for that given Lagrangian and field. The internal symmetry of $\mathcal{L}$ is a discrete symmetry: $\phi \to -\phi$ which would be broken by $v \neq 0$.

Already classically one has SSB. Let us first notice that for the Lagrangian in (3.3) we have from the condition (3.4):

$$m^2\phi + \frac{\lambda}{3!}\phi^3 = \phi\left(m^2 + \frac{\lambda}{3!}\phi^2\right) = 0\,, \tag{3.5}$$

which for a real field has *only* the solution $\phi = 0$, then $v = 0$ and we have no SSB.
However if one takes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V[\phi(x)]\,, \quad V[\phi(x)] = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\,, \tag{3.6}$$

in this case we obtain

$$\frac{\partial V[\phi]}{\partial\phi} = 0 = \phi\left(-\mu^2 + \frac{\lambda}{3!}\phi^2\right)\,,$$

that brings to

$$\begin{cases} \phi = 0 & \text{(unstable vacuum)} \\ \phi = \pm\sqrt{\frac{6}{\lambda}}\mu = \pm v\,. \end{cases} \tag{3.7}$$

(see figure 3.3) and we notice that the vacua $\phi \propto 1/\sqrt{\lambda}$ appear that signal a non perturbative SSB. We have two degenerate vacua, $\pm v$, and under the symmetry $\phi \to -\phi$ one

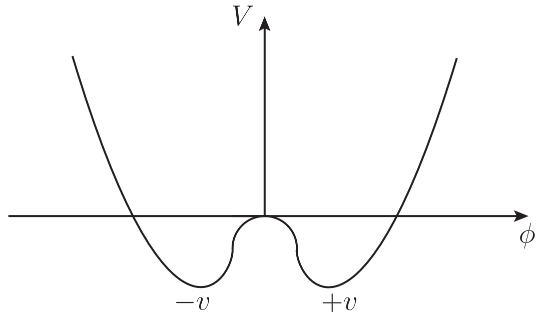


Figure 3.3: Quartic potential and vacua.

vacuum changes to the other one.
Typically one would like to rewrite the theory in terms of shifted fields

$$\phi(x) = v + \sigma(x)\,, \tag{3.8}$$

so that the potential has a minimum at $\sigma(x) = 0$ (it is a stable vacuum). If one rewrites (3.6) in terms of (3.8) finds (dropping constants terms)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma(x))^2 - \frac{1}{2}(2\mu^2)\sigma^2(x) - \sqrt{\frac{\lambda}{6}}\mu\sigma^3(x) - \frac{\lambda}{4!}\sigma^4(x)\,, \tag{3.9}$$

where the term linear in $\sigma(x)$ vanishes (the minimum of the potential is at $\sigma(x) = 0$).

The Lagrangian (3.9) describes a scalar field of mass $\sqrt{2}\mu$ with σ^3 and σ^4 interactions. *The symmetry $\phi \to -\phi$ is no longer apparent*, its only manifestation is in the relation among the three coefficients in (3.9) which depend in a special way on only two parameters.
This is the simplest example of broken symmetry (for a discrete symmetry). For a continuous symmetry the spectrum content is different: it may lead to generation of a massless field (Goldostone boson), or it would give mass to the gauge bosons in the case of a gauge theory. (See exercises.)
This is anyway a classical description. The vacuum expectation value $\langle 0|\phi(x)|0\rangle$ of the QFT will not be necessarily the same of the classical field theory.

In QFT the SSB is given by the condition

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} \neq 0\,, \tag{3.10}$$

and consequently

$$\frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle} = \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = v\,. \tag{3.11}$$

This corresponds to the following condition on the effective action:

$$\left.\frac{\delta \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)}\right|_{\phi_{\rm cl}=v} = 0\,, \tag{3.12}$$

because now when $J = 0 \Rightarrow \phi_{\rm cl} = \langle\phi\rangle = v$ and viceversa.
(Recall the definitions

$$\phi_{\rm cl} = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi|0\rangle_J}{\langle 0|0\rangle_J}\,,$$

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = -J(x)\,,$$

useful to write the previous equations.)
So we can use the effective action to find the v.e.v. of the quantum field. One would like to work with the shifted fields as we have done before in the example: $\bar{\phi} = \phi - v$ and we have $\langle\bar{\phi}\rangle = 0$. *It can be demonstrated that the proper vertex functions in a theory with SSB can be given in terms of proper vertex functions calculated in the symmetric mode* (Renormalization is the same).

3.3 Effective potential

Recall the general definition for the effective action as follows

$$\Gamma[\phi_{\rm cl}] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\Gamma^{(n)}(x_1,\,\ldots,x_n)\phi_{\rm cl}(x_1)\cdots\phi_{\rm cl}(x_n)\,, \tag{3.13}$$

we may expand it in powers of momentum about zero momentum, in position space we have

$$\Gamma[\phi_{\rm cl}] = \int d^4x\left[-U(\phi_{\rm cl}) + \frac{A(\phi_{\rm cl})}{2}\partial_\mu\phi_{\rm cl}\partial^\mu\phi_{\rm cl} + \cdots\right]\,, \tag{3.14}$$

where $U(\phi_{\rm cl})$ and $A(\phi_{\rm cl})$ are some functions of $\phi_{\rm cl}$. In the case of classical field which is constant in both space and time, $\phi_{\rm cl} = \varphi = v$, only the term in U survives, hence the effective potential is defined by

$$\Gamma[\phi_{\rm cl}(x) = \varphi] = -(2\pi)^4\delta(0)U(\varphi)\,. \tag{3.15}$$

(Note that $U(\varphi)$ is a function of φ, not a functional.)

Then the condition

$$\left.\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}\right|_{\phi_{\rm cl}=v} = 0 \;\Rightarrow\; \left.\frac{dU}{d\varphi}\right|_{\varphi=v} = 0\,. \tag{3.16}$$

This equation is more useful as $U(\varphi)$ can be calculated systematically in a perturbation expansion.

Recalling eq. (2.80), one can write $\Gamma[\varphi]$ as

$$\begin{aligned}\Gamma[\varphi] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\ldots\int d^4x_n\Gamma^{(n)}(x_1,\,\ldots,x_n)\varphi^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\varphi^n\int d^4x_1\ldots\int d^4x_n\Gamma^{(n)}(x_1,\,\ldots,x_n) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\varphi^n\tilde{\Gamma}^{(n)}(0,\cdots,0)\,(2\pi)^4\delta^4(0)\,,\end{aligned} \tag{3.17}$$

where $\tilde{\Gamma}^{(n)}(0,\cdots,0)$ is the Fourier transform of $\Gamma^{(n)}(x_1,\,\ldots,x_n)$ at $p_i = 0$. Comparing equations (3.15) and (3.17), one can see that

$$U(\varphi) = -\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\Gamma}^{(n)}(0,\cdots,0)\,\varphi^n, \tag{3.18}$$

so that

$$-\frac{d^nU(\varphi)}{d\varphi^n}\bigg|_{\varphi=0} = \tilde{\Gamma}^{(n)}(0)\,, \tag{3.19}$$

where the proper vertex function $\tilde{\Gamma}^{(n)}(0)$ is the sum of all 1PI amputated diagrams for the original (not shifted) field of the Lagrangian with n vanishing external momenta.

The knowledge of the effective potential means that we know the structure of the SSB.

Notice:
the calculation of the effective potential in (3.19) involves a double sum in perturbation theory: over all $\tilde{\Gamma}^{(n)}$ for any n and for each $\tilde{\Gamma}^{(n)}$ there is an expansion in terms of the coupling constant. It will be convenient to organize the double expansion in terms of the number of loops. This is the so-called loop expansion, which is an expansion according to the increasing number of the independent loops of connected Feynman diagrams.

The effective potential coincides at leading order (tree level) with the classical potential. In fact, at tree level, the only non vanishing vertex functions are (in momentum space):

$$\begin{cases} \tilde{\Gamma}^{(2)} = p^2 - m^2\,, \\ \tilde{\Gamma}^{(4)} = -\lambda\,, \end{cases} \tag{3.20}$$

then from eq. (3.18) we have

$$U(\varphi) = (-p^2 + m^2)\frac{\varphi^2}{2} + \frac{\lambda}{4!}\varphi^4 = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 = V_{\rm cl}(\varphi)\,, \tag{3.21}$$

where $p^2\varphi^2 = 0$ for φ constant.
Useful physical quantities can be obtained directly from the effective potential:

$$\begin{cases} \frac{dU(\varphi)}{d\varphi}\Big|_{\varphi=v} = 0\,, \quad \text{condition for the v.e.v.,} \\ \\ \frac{d^2U(\varphi)}{d\varphi^2}\Big|_{\varphi=v} = m_R^2\,, \quad \text{renormalized mass,} \\ \\ \frac{d^4U(\varphi)}{d\varphi^4}\Big|_{\varphi=v} = \lambda_R\,, \quad \text{renormalized coupling constant}\,, \end{cases} \tag{3.22}$$

where one has renormalized the $\tilde{\Gamma}^{(n)}$ then the derivatives in eq. (3.22) gives the renormalized mass and coupling constant. Note that if $\varphi(x)$ were to vary in space and time then we should calculate the v.e.v. from the more complicated equation

$$\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}\bigg|_{\phi_{\rm cl}=\varphi(x)} = 0\,. \tag{3.23}$$

Since we should have $m_R^2 \geq 0$, the first two conditions say that $U(\varphi)$ has a minimum in $\varphi = v$, in the same way the classical potential had a minimum in v. It can be shown that $U(\varphi)$ *gives the energy density of the vacuum.*
We can also work in terms of the shifted field $\sigma(x) = \phi(x) - v$, then the $\Gamma_\sigma^{(n)}$, defined

with respect to $\sigma(x)$, are linear combination of $\Gamma_\phi^{(n)}$ and may be obtained by the shifted version of (3.18):

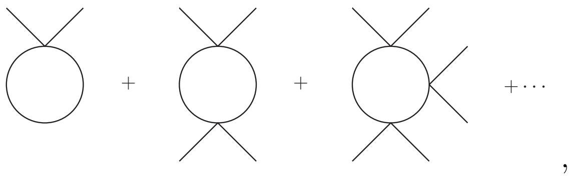
$$U(\phi - v) = -\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_\sigma^{(n)}(0, \cdots, 0)(\phi - v)^n . \tag{3.24}$$

One can show that, since all the divergences of the theory have been taken care with counterterms before applying the renormalization condition (3.22), *no additional divergences appear in a theory with SSB compared to the unbroken theory.* So we conclude that *the divergence structure of the theory is not affected by the occurrence of SSB.*

A perfect example for this conclusion is the Weinberg-Salam model (the electroweak part of the Standard Model). This a gauge quantum field theory and is renormalizable, although we know that massive vector particles (such as W^+) would destroy renormalizability of an *ordinary* gauge theory. The combination of having a renormalizable gauge theory and having massive vector particles is possible because the masses are originated via a SSB mechanism. Then the renormalization is still constrained by the underlying gauge structure, as we have explained above.

3.4 Loop expansion of the effective potential

The loop expansion is an expansion according to the increasing number of the independent loops of connected Feynman diagrams. We already saw that the expansion in powers of $\hbar$ is a reordering of perturbation theory according to the number of loops of the Feynman diagrams. So we will restore $\hbar$ in our formalism in order to facilitate the loop expansion. In the loop expansion, the following diagrams contribute to $U(\phi)$ at one loop



i.e., all one-loop diagrams with an even number of external lines having zero momenta. One can calculate the one-loop contribution for example using the saddle point evaluation of the path integral. You have done this in the exercise sheet 4. Let us recall the main points of the exercise you have already done.

First of all, given an integral of the form

$$I = \int dx\, e^{if(x)} , \tag{3.25}$$

with $f(x)$ stationary at $x = x_0$, *i.e.*, $df/dx|_{x=x_0} = 0$, we have

$$f(x) = f(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \Rightarrow I \simeq e^{if(x_0)} \int dx\, e^{\frac{i}{2}(x-x_0)^2 f''(x_0)} , \tag{3.26}$$

so that the integral is Gaussian and can be evaluated using the $(i\epsilon)$ trick. We apply the saddle point approximation to

$$e^{\frac{i}{\hbar}W[J]} = N \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\left[S[\phi] + \hbar \int d^4x J(x)\phi(x)\right]\right\} , \tag{3.27}$$

with

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) \; ; \quad V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \,, \tag{3.28}$$

and we expand (3.27) around $\phi_0(x)$ solution of the equation

$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} = -\hbar J(x) \,, \tag{3.29}$$

where ϕ_0 is a solution of the classical equation in presence of $J(x)$. We expand around ϕ_0 the action as follows

$$S[\phi+\phi_0] = S[\phi_0] + \int d^4x\phi(x) \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} + \frac{1}{2}\int d^4x d^4y \phi(x)\phi(y) \left.\frac{\delta^2 S}{\delta\phi(x)\phi(y)}\right|_{\phi=\phi_0} + S_2[\phi,\phi_0] \,, \tag{3.30}$$

and we also add the useful equation

$$\left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_0} = -\left(\partial^2 + V''(\phi_0)\right)\delta^4(x-y) \,. \tag{3.31}$$

Then substituting eq. (3.29) and (3.31) into (3.30) we have

$$S[\phi+\phi_0] = S[\phi_0] - \hbar\int d^4x\phi(x)J(x) - \frac{1}{2}\int d^4x\phi(x)\left(\partial^2 + V''(\phi_0)\right)\phi(x) + S_2[\phi,\phi_0] \,. \tag{3.32}$$

Inserting (3.32) back into (3.27) we have

$$e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}\left\{S[\phi_0] + \hbar\int d^4x\phi_0 J\right\}} N \int \mathcal{D}\phi e^{-\frac{i}{2\hbar}\int d^4x\phi(x)\left(\partial^2 + V''(\phi_0)\right)\phi(x)} \,. \tag{3.33}$$

We then scale $\phi \to \sqrt{\hbar}\phi$ to eliminate $\hbar$ and we obtain for the second factor

$$N\int \mathcal{D}\phi e^{-\frac{i}{2}\int d^4x\phi(x)\left(\partial^2 + V''(\phi_0)\right)\phi(x)} = N\left[\det\left(\partial^2 + V''(\phi_0)\right)\right]^{-\frac{1}{2}} \,, \tag{3.34}$$

where in the last step we used Gaussian integration. By using

$$\det A = e^{\mathrm{tr}\ln A} \Rightarrow (\det A)^{-\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{tr}\ln A} \simeq 1 - \frac{1}{2}\mathrm{tr}\ln A \,, \tag{3.35}$$

we obtain finally:

$$W[J] = S[\phi_0] + \hbar\int d^4x J(x)\phi_0(x) + i\frac{\hbar}{2}\mathrm{tr}\ln\left(\partial^2 + V''(\phi_0)\right) - i\hbar\ln N \,, \tag{3.36}$$

where we have discarded the $S_2[\phi,\phi_0]$ that gives contributions in $\hbar^2$. Imposing the condition $W[J]=0$, one can always determine the constant $-i\hbar\ln N$.
We can now obtain the effective action $\Gamma[\phi_{\rm cl}]$ at one loop. However we must express $S[\phi_0]$ as a functional of $\phi_{\rm cl}$. We know that $\phi_{\rm cl} = \phi_0$ in the tree level approximation, so that $\phi_{\rm cl} - \phi_0 \simeq \mathcal{O}(\hbar)$ and

$$\phi_{\rm cl} = \frac{\delta W[J]}{\hbar\delta J(x)} = \phi_0 + \phi_1(x) \,, \quad \phi_1(x) \sim \mathcal{O}(\hbar) \,. \tag{3.37}$$

So we can calculate:

$$S[\phi_0] = S[\phi_{\rm cl} - \phi_1] = S[\phi_{\rm cl}] - \int d^4x \phi_1 \underbrace{\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0}}_{-\hbar J(x)} + \mathcal{O}(\hbar^2)\,, \tag{3.38}$$

then we use

$$\Gamma[\phi_{\rm cl}] = W[J] - \hbar \int d^4x J(x)\phi_{\rm cl}(x) = S[\phi_{\rm cl}] + i\frac{\hbar}{2}{\rm tr}\ln\left(\partial^2 + V''(\phi_0)\right) - i\hbar \ln N + \mathcal{O}(\hbar^2) \tag{3.39}$$

and substituting (3.36) and (3.38) and considering that at the order we are interested J is cancelled (so that we do not need to invert (3.37)). Now we put $\phi_{\rm cl} = {\rm const} = v = \varphi$. We know that

$$\begin{cases} \Gamma[\varphi] = -\Omega U(\varphi)\,, \\ S[\varphi] = \int d^4x \mathcal{L}|_{\phi(x)=\varphi} = -\Omega V(\varphi)\,, \quad \Omega = (2\pi)^4\delta(0) \end{cases}$$

so that we can write

$$U(\varphi) = V(\varphi) - i\frac{\hbar}{2}\Omega^{-1}{\rm tr}\left(\partial^2 + V''(\varphi)\right) + \mathcal{O}(\hbar^2)\,. \tag{3.40}$$

Since the trace of an operator is the sum (integral) over its eigenvalues, eq. (3.40) reads

$$U(\varphi) = V(\varphi) + \frac{\hbar}{2}\int \frac{d^4k_E}{(2\pi)^4}\ln\left(\frac{k_E^2 + V''(\varphi)}{k_E^2 + V''(0)}\right) + \mathcal{O}(\hbar^2)\,. \tag{3.41}$$

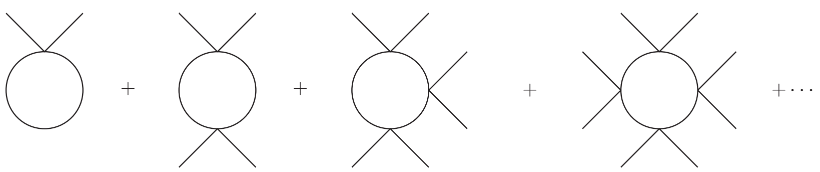
Note that the above relation is written in momentum space after going into Euclidean coordinates. (The factor i is absorbed into the time component once Euclidean coordinates are used.)

Clearly the one loop contribution to U is divergent and has to be renormalized. In particular for ϕ^4 theory, we have

$$U(\varphi) = V(\varphi) + \frac{\hbar}{2}\int \frac{d^4k_E}{(2\pi)^4}\ln\left(\frac{k_E^2 + m^2 + \lambda\frac{\varphi^2}{2}}{k_E^2 + m^2}\right)\,. \tag{3.42}$$

Now consider the following question. When $m^2 > 0$ the vacuum is non-degenerate, and when $m^2 < 0$ it is degenerate. What sort of vacuum do we have when $m^2 = 0$? Imposing the condition that the renormalized mass is still zero (Eq. (3.22)), it turns out that the minimum of the effective potential occurs for $\langle\phi\rangle \neq 0$, so there is spontaneous symmetry breaking, induced by radiative correction (by quantum effects at one loop level).

One can get the same result by direct calculation by summing all the contribution



The vertex function with vanishing external momenta is given by (we consider the case $m^2 > 0$ no SSB)

$$\Gamma^{(2n)}(0, \dots, 0) = iS_n \int \frac{d^4k}{(2\pi)^4} \left((-i\lambda) \frac{i}{k^2 - m^2 + i\epsilon} \right)^n , \tag{3.43}$$

with S_n is a symmetry factor that reads:

$$S_n = \frac{(2n)!}{2^n 2n} . \tag{3.44}$$

Let us comment on the symmetry factor:

- there are $(2n)!$ ways to distribute $2n$ particles to the external lines of the diagram,
- there are 2^n interchanges of any two external lines at a given vertex that do not lead to new contributions,
- there are $2n$ reflections and rotations of n vertices of the ring that do not lead to new contributions.

Therefore the tree level and one loop contributions are, from eq. (3.18)

$$\begin{aligned} U(\varphi) &= V(\varphi) + i \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left[\frac{\lambda}{2} \frac{\varphi^2}{k^2 - m^2 + i\epsilon} \right]^n \\ &= \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left[1 + \frac{\lambda}{2} \frac{\varphi^2}{k^2 - m^2 + i\epsilon} \right] , \end{aligned} \tag{3.45}$$

which is what we already obtained in eq. (3.42).

4 Scattering amplitudes

Let us briefly recall that any free classical field can be written as follows

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a^*_{\mathbf{p}}e^{ipx})\,. \tag{4.1}$$

We notice that the initial state, incoming, ϕ_{in} and the final state, outgoing, ϕ_{out} are special case of the mode expansion for the field $\phi_0(x)$. The initial state ($t \to -\infty$) is associated with the positive frequency piece of ϕ_0:

$$\phi_{\text{in}}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}e^{-ipx}\,,$$

and the final outgoing state ($t \to +\infty$) is associtaed with the negative frequency part of ϕ_0:

$$\phi_{\text{out}}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}a^*_{\mathbf{p}}e^{ipx}\,.$$

This separation can be achieved by assigning a small negative imaginary part to $p_0 \to p_0 - i\epsilon$, so that $e^{ip_0t} \to e^{ip_0t}e^{\epsilon t}$, and then only the $e^{-ix\cdot p}$ piece survives for $t \to -\infty$. (Viceversa for the other limit $t \to +\infty$).

We now know how to calculate the Green's functions in the interacting theory, in a perturbation theory in λ. We want now to calculate scattering amplitudes in terms of Green's functions.

We have already obtained in the Introduction to QFT the relation between the scattering amplitudes and the Green's functions of a given theory. There we proved and used the LSZ theorem (Lehman,Symanzik and Zimmermann). The full derivation will be not repeated here (if you want to see it look at the notes on Relativity, Particles and fields).

Let me just recall what were the main points. We know how to calculate in the path integral formalism transition amplitudes of the form $\langle\phi''(t'',\mathbf{x})|\phi'(t',\mathbf{x})\rangle_J$, where we have a real scalar field that achieves the configurations $\phi''(t'',\mathbf{x})$ at $t = t''$ and $\phi'(t',\mathbf{x})$ at $t = t'$ respectively, in presence of a source $J(x)$. Then to calculate S-matrix elements we need to calculate the transition amplitude by performing the functional integral over all field configurations $\phi(t,\mathbf{x})$ having free asymptotic behaviour as follows:

$$\begin{cases} \phi(t,\mathbf{x}) \to \phi_{\text{in}}(t,\mathbf{x}) & \text{for } t \to -\infty\,, \\ \phi(t,\mathbf{x}) \to \phi_{\text{out}}(t,\mathbf{x}) & \text{for } t \to +\infty\,, \end{cases} \tag{4.2}$$

where $\phi_{\text{in}}(t,\mathbf{x})$ and $\phi_{\text{out}}(t,\mathbf{x})$ satisfy the free field equations:

$$\left(\partial_x^2 + m^2\right)\phi_{\text{in,out}}(x) = 0\,. \tag{4.3}$$

No we would like to determine the functional that generates S-matrix elements. It is clear that we have to use in the calculations ($\hbar = 1$):

$$Z[J] = N\int \mathcal{D}\phi e^{i\int d^4x\,(\mathcal{L}+J\phi)} \tag{4.4}$$

We define

$$A[J,\phi_0]=\int \mathcal{D}\phi e^{i\int d^4x\,(\mathcal{L}+J\phi)}\,, \tag{4.5}$$

where ϕ_0 means that $t\to\pm\infty$, $\phi(x)=\phi_0(x)$, or precisely

$$\begin{cases}\phi_0(x)\sim\phi_{\rm in}(x) & \text{for } t\to-\infty\,,\\ \phi_0(x)\sim\phi_{\rm out}(x) & \text{for } t\to+\infty\,,\end{cases} \tag{4.6}$$

so that now the integration is over all field configurations having asymptotic behaviour in (4.6). We want to determine $S[J,\phi_0]$ in terms of $Z[J]$. So we write

$$\begin{cases}A[J,\phi_0]=\exp\left\{i\int d^4x\mathcal{L}_I\left(\frac{\delta}{i\delta J(x)}\right)\right\}A_0[J,\phi_0]\,,\\ \\ A_0[J,\phi_0]=\int\mathcal{D}\phi\exp\left\{i\int d^4x\,(\mathcal{L}_0+J\phi)\right\}=\int\mathcal{D}\phi\exp\left\{iS_0[\phi]+i\int d^4x\,(J\phi)\right\}\,,\end{cases} \tag{4.7}$$

where the integration is still over fields having the asymptotic behaviour in (4.6). Now we can change the functional integration variable from ϕ to $\tilde\phi$ so that

$$\begin{cases}\phi(x)=\tilde\phi(x)+\phi_0(x)\,,\\ \tilde\phi\to 0 \text{ as } t\to\pm\infty\,.\end{cases} \tag{4.8}$$

Now we can use the fact that ϕ_0 satisfies the equation of motions and we write

$$\begin{aligned}S_0[\phi]+\int d^4xJ\phi=\int d^4x\Big[&\frac{1}{2}(\partial_\mu\tilde\phi)(\partial^\mu\tilde\phi)-\frac{1}{2}m^2\tilde\phi^2+J(\tilde\phi+\phi_0)\\ &+\frac{1}{2}(\partial_\mu\phi_0)(\partial^\mu\phi_0)-\frac{1}{2}m^2\phi_0^2+\partial^\mu\tilde\phi\partial_\mu\phi_0-m^2\tilde\phi\phi_0\Big]\,,\end{aligned} \tag{4.9}$$

and integrating by parts, we get for the first and third term in the second line of (4.9)

$$-\tilde\phi\partial^\mu\partial_\mu\phi_0-\frac{1}{2}\phi_0\partial^\mu\partial_\mu\phi_0\,.$$

Then because ϕ_0 is the free field we have $(\partial^2+m^2)\phi_0=0$ and then

$$-\tilde\phi(\partial^2+m^2)\phi_0-\frac{1}{2}\phi_0(\partial^2+m^2)\phi_0=0\,,$$

so that we are left with the following expression for (4.9)

$$S[\tilde\phi]+\int d^4x(\tilde\phi+\phi_0)J\,.$$

Now we obtain for $A_0[J,\phi_0]$ the following expression

$$A_0[J,\phi_0]=\underbrace{\int\mathcal{D}\tilde\phi\,\exp\left\{i\left(S_0[\tilde\phi]+\int d^4xJ\tilde\phi\right\}\right)}_{Z_{\rm free}[J]}\exp\left\{i\int d^4xJ(x)\phi_0(x)\right\}\,. \tag{4.10}$$

Now the functional integral is over paths that satisfy (4.8) which is a special case of fixed boundary condition

$$\phi''(t'',\mathbf{x})=\phi'(t',\mathbf{x})=0\,. \tag{4.11}$$

Then the first factor in (4.10) is $Z_{\text{free}}[J]$. Moreover we can prove that

$$(\partial_x^2 + m^2)\frac{\delta Z_{\text{free}}[J]}{\delta J(x)} = iJ(x)Z_{\text{free}}[J]\,. \tag{4.12}$$

Let us prove it. We start with

$$Z_{\text{free}}[J] = \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y)J(y)\right\}\,. \tag{4.13}$$

The we use the results

$$\frac{Z_{\text{free}}[J]}{\delta J(x)} = \left[-\int d^4y D_F(x-y)J(y)\right] Z_{\text{free}}[J]\,, \tag{4.14}$$

$$(\partial_x^2 + m^2)D_F(x-y) = -i\delta^4(x-y)\,, \tag{4.15}$$

that combined together allows us to write

$$\begin{aligned}&(\partial_x^2 + m^2)\left[-\int d^4y D_F(x-y)J(y)\right] Z_{\text{free}}[J] \\ &= -\int d^4y J(y)(-i)\delta^4(x-y)Z_{\text{free}}[J] = iJ(x)Z_{\text{free}}[J]\,.\end{aligned} \tag{4.16}$$

Hence, using eq. (4.12), eq. (4.10) becomes

$$A_0[J,\phi_0] = \exp\left\{\int d^4x\phi_0(x)(\partial_x^2 + m^2)\frac{\delta}{\delta J(x)}\right\} Z_{\text{free}}\,, \tag{4.17}$$

and then using eq. (4.7) and (4.17) and interchanging the order of the functional derivatives:

$$\begin{aligned}A[J,\phi_0] &= \exp\left\{\int d^4x\phi_0(x)(\partial_x^2 + m^2)\frac{\delta}{\delta J(x)}\right\}\exp\left\{i\int d^4y\mathcal{L}_I\left(\frac{\delta}{\delta J(y)}\right)\right\} Z_{\text{free}}[J] \\ &= \exp\left\{\int d^4x\phi_0(x)(\partial_x^2 + m^2)\frac{\delta}{\delta J(x)}\right\} Z[J]\,.\end{aligned} \tag{4.18}$$

This gives *the asymptotic state to asymptotic state transition amplitude in presence of a source J(x).* If there is no source, the quantity of interest is $A[0,\phi_0]$.

From eq. (4.18) we can say the following: ϕ_0 is a free field, its Fourier transform satisfies the mass shell condition $k^2 = m^2$. If we integrate by part in (4.18), $(\partial_x^2 + m^2)$ goes over ϕ_0 and then produce a factor $m^2 - k^2$ for each Fourier component k. Unless the contribution from $Z[J]$ has a compensating $(k^2 - m^2)^{-1}$, its contribution to $A[J,\phi_0]$ will vanish since $m^2 - k^2 = 0$ for all Fourier components of ϕ_0. Thus the effect of the propagator in (4.10) is to project *off-mass-shell contributions to zero in $Z[J]$ and retain only on shell contribution as required.* Setting $J = 0$ ensures that all external lines of $Z[J]$ are subjected to this treatment.
So if we put $J = 0$ we have the functional $S[\phi_0]$ that generates scattering amplitudes

$$A[\phi_0] = \exp\left\{\int d^4x\phi_0(x)(\partial_x^2 + m^2)\frac{\delta}{\delta J(x)}\right\} Z[J]\Big|_{J=0}\,, \tag{4.19}$$

and by expanding the exponential and denoting:

$$k_{x_j} = \partial_{x_j}^2 + m^2 \tag{4.20}$$

we have:

$$A[\phi_0] = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int d^4x \ldots d^4x_n \, \phi_0(x_1) \ldots \phi_0(x_n) k_{x_1} \ldots k_{x_n} G^{(n)}(x_1, \ldots, x_n) \,. \tag{4.21}$$

So each external line of $G^{(n)}(x_1, \ldots, x_n)$ has attached a Klein-Gordon operator k_{x_j} and a free field $\phi_0(x_j)$. In scattering we deal with n incoming or outgoing free particles which are momentum eigenstates. Inserting the momentum space form of the Green's functions we find:

$$\begin{aligned} A[\phi_0] = & \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + \cdots + p_n) \\ & (m^2 - p_1^2) \cdots (m^2 - p_n^2) \tilde{G}^{(n)}(p_1, \ldots, p_n) \\ & \int d^4x_1 \cdots \int d^4x_n \, e^{-ip_1 \cdot x_1 - \ldots - ip_n \cdot x_n} \phi_0(x_1) \ldots \phi_0(x_n) \,. \end{aligned} \tag{4.22}$$

On the other hand

$$\int d^4x e^{-ip \cdot x} \phi_0(x) = \int d^3k \frac{2\pi}{\sqrt{2E_k}} \left[a_{\mathbf{k}} \delta^4(p+k) + a_{\mathbf{k}}^* \delta^4(p-k) \right] , \tag{4.23}$$

where we have inserted the Fourier decomposition for the ϕ_0 field to get such result, and with $E_k = \sqrt{\mathbf{k}^2 + m^2}$, so that the only momentum that can contribute to $A[\phi_0]$ must satisfy $p_j = \pm k$ (the momentum of the jth particle) and since

$$k^2 = m^2 \Rightarrow p_j^2 = m^2 \,. \tag{4.24}$$

When (4.24) is satisfied only one of the two terms in (4.23) can contribute if $(p_j)_0 < 0$, *i.e.*, $(p_j)_0 = -k_0$, only the first term contributes, conversely for $(p_j)_0 > 0$ only the second term contributes. This reflects the fact that in physical processes some particles are associated with incoming particles and some with outgoing particles. So the only Green's functions that can contribute are those with external lines *on the mass shell*.

Now consider a scattering process with m particles in the initial state (incoming momenta $q_1 \ldots q_m$) and $n - m$ particles in the final state (outgoing momenta $q_{m+1} \ldots q_n$). All the momenta are physical (= of physical particles)

$$\begin{cases} q_i = m^2 \,, \quad i = 1, \ldots, n \\ q_1 + \ldots + q_m = q_{m+1} + \ldots + q_n \,. \end{cases} \tag{4.25}$$

The scattering amplitude is given by the part of $A[\phi_0]$ involving m factors $a_{\mathbf{q}_1} \ldots a_{\mathbf{q}_m}$ and $n - m$ factors $a_{\mathbf{q}_{m+1}}^* \ldots a_{\mathbf{q}_n}^*$

$$S_{\mathrm{fi}} = [\rho(q_1) \ldots \rho(q_n)]^{-1} \left. \frac{\delta^n S[\phi_0]}{\delta a_{\mathbf{q}_1} \ldots \delta a_{\mathbf{q}_m} \delta a_{\mathbf{q}_{m+1}}^* \ldots \delta a_{\mathbf{q}_n}^*} \right|_{a=a^*=0} , \tag{4.26}$$

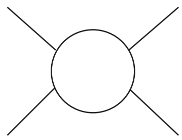
where

$$\rho(q) = (2\pi)^{-3} (2q_0)^{-\frac{1}{2}} \,, \tag{4.27}$$

is the covariant momentum integration weight function. The reason to set $a = a^* = 0$ at eq. (4.26) is to ensure that only G^n contributes to the n-particle process. Then, substituting (4.22) in (4.26) and performing the differentiation we obtain

$$S_{\rm fi} = (2\pi)^4 \delta^4(-q_1 - \, ... \, - q_m + q_{m+1} + \, ... \, + q_n) M_{\rm fi} \, ,$$
$$M_{\rm fi} = (-i)^n (q_1^2 - m^2) \cdots (q_n^2 - m^2) \, \tilde{G}^{(n)}(-q_1, \, ... \, , -q_m, +q_{m+1}, \, ... \, , +q_n) \, . \quad (4.28)$$

$\tilde{G}^{(n)}$ is the full-complete Green's function including disconnected and 1PR pieces, as well 1PI and connected. The Green's functions have propagators on the external lines that would diverge on shell, but the factors $-i(q_i^2 - m^2)$ cancel the diverging parts. So the S-matrix elements, S_{fi}, are obtained from the complete Green functions by defining the external line propagators and supplying an overall delta function on the momenta of incoming and ongoing particles. Then the Feynman rules for S-matrix elements are similar to those of 1PI because one has to amputate the external propagators, however in the case of S-matrix the external lines have the condition to be on shell and all contributing disconnected and 1PR diagrams ought to be considered as well together with IPI diagrams. In some cases most of the disconnected diagrams do not contribute to the actual scattering process. For $2 \to 2$ processes



energy momentum conservation ensures that in this case all disconnected diagrams contribute only when the initial and final state are identical which is the case in which there is no scattering (the 1 part in $S = 1 - iT$).
The scattering cross section, by definition, measures cases in which the initial and final state differ $\Rightarrow$ disconnected diagrams do not contribute to the two particle cross section. In exercise (3.2), we have seen the calculation of scattering cross section of a $2 \to 2$ problem.

5 Path Integrals for fermions

We have seen that the path integral quantization method involves only classical fields and the theory is formulated in terms of the generating functional $Z[J]$ and in terms of the Green's functions $G^{(n)}$ that can be obtained from it. For a scalar field we had:

$$G^{(n)}(x_1, \ldots, x_n) = N \int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]} = \langle 0|T\hat{\phi}(x_1)...\hat{\phi}(x_n)|0\rangle \,. \tag{5.1}$$

However we know from the spin-statistics theorem that fields of spin 1/2 should be described by anticommutating variables. In fact in canonical quantization we assumed that

$$\begin{aligned} &\left\{\hat{\psi}(x), \hat{\psi}(y)\right\}_{x_0=y_0} = 0\,, \quad \left\{\hat{\psi}_i(x), \hat{\psi}^\dagger_j(y)\right\}_{x_0=y_0} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{ij}\,, \\ &\left\{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\right\}_{x_0=y_0} = 0\,. \end{aligned} \tag{5.2}$$

Actually fermion fields anticommute at all times. In the canonical approach to field theory the $\psi(x)$ are regarded as operators and they are a set of annticommuting operators, so that for the two-point Green's function, for example, it holds that

$$\langle 0|T\hat{\psi}(x)\hat{\bar{\psi}}(y)|0\rangle = -\langle 0|T\hat{\bar{\psi}}(y)\hat{\psi}(x)|0\rangle \tag{5.3}$$

In the path integral we use only classical fields, which are, so far, commuting numbers. Such a formalism cannot support the Pauli principle. For example, with the formalism developed so far, a particle could propagate to a point in the coordinate space which is already occupied. In nature, however, this propagation is Pauli-forbidden for fermions. In order to extent our path integral formalism to include fermions, we use anticommutating quantities as mathematical objects corresponded to the fermionic fields. These mathematical objects are good candidates to ensure that the Pauli principle is not violated.

5.1 Anticommuting numbers - Grassman algebra

This was a mathematical construction dating back to 1855 where it appeared in a paper by Hermann and Grassman on linear algebra.

A Grassman variable η is defined by:

$$\{\eta, \eta\} = 0\,, \tag{5.4}$$

from which it follows $\eta^2 = 0$ and thus any function of η can be written as:

$$f(\eta) = f_0 + \eta f_1\,. \tag{5.5}$$

Note that the coefficient f_0 and f_1 can be ordinary numbers or Grassman numbers. When f_1 is a number and f_0 is a Grassman number, $f(\eta)$ is Grassman variable, *i.e.*, $\{f(\eta), f(\eta)\} = 0$. And when f_1 is a Grassman variable and f_0 is a number, $f(\eta)$ is an ordinary number , *i.e.*, $f(\eta)$ like any other ordinary number obeys a commuting algebra.

Because they are anticommuting variables there are two types of differentiations, namely left and right:

$$\frac{d}{d\eta}\eta = \eta\frac{\overleftarrow{d}}{d\eta} = 1\,. \tag{5.6}$$

When f_1 is a Grassman variable then

$$\frac{d}{d\eta}f(\eta) = f_1 = -f(\eta)\frac{\overleftarrow{d}}{d\eta}\,, \tag{5.7}$$

because f_1 anticommutes with η. We have

$$\begin{aligned}&\left\{\frac{d}{d\eta},\eta\right\}f(\eta) = \frac{d}{d\eta}(\eta f(\eta)) + \eta\frac{d}{d\eta}f(\eta) = +f(\eta) - \eta\frac{d}{d\eta}f(\eta) + \eta\frac{d}{d\eta}f(\eta) \\ &\Rightarrow \left\{\frac{d}{d\eta},\eta\right\} = 1\,.\end{aligned} \tag{5.8}$$

Note that for ordinary variables we have $[\frac{d}{dx}, x] = 1$, so for Grassman variables we have a similar relation but the commutator is replaced with anticommutator. If we have more than one Grassman variable we can define the generators of an n-dimensional Grassman algebra, they obey :

$$\{\eta_i, \eta_j\} = 0\,, \quad i,j = 1,\cdots,n\,. \tag{5.9}$$

The expansion of any f over these variables contains only a finite number of terms due to the fact that $\eta_i^2 = 0$. Then we have for example:

$$\begin{cases}\frac{\vec{\partial}}{\partial\eta^i}(\eta_1\eta_2) = \delta_{i1}\eta_2 - \delta_{i2}\eta_1\,, \\ (\eta_1\eta_2)\frac{\overleftarrow{\partial}}{\partial\eta^i} = -\delta_{i1}\eta_2 + \delta_{i2}\eta_1\,,\end{cases} \tag{5.10}$$

so that

$$\left\{\frac{\partial}{\partial\eta^i},\eta_j\right\} = \left(\frac{\partial}{\partial\eta^i}\eta_j + \eta_j\frac{\partial}{\partial\eta^i}\right) = \delta_{ij}\,. \tag{5.11}$$

We also have

$$\frac{\partial}{\partial\eta_1}(\eta_1\eta_2) = -\eta_1\frac{\partial}{\partial\eta_1}\eta_2 + \eta_2 = \eta_2\,. \tag{5.12}$$

One can also verify that

$$\left\{\frac{\partial}{\partial\eta^i},\frac{\partial}{\partial\eta^j}\right\} = 0 \Rightarrow \frac{\partial^2}{\partial\eta_1\partial\eta_2} = -\frac{\partial^2}{\partial\eta_2\partial\eta_1}\,. \tag{5.13}$$

Then we have $(\partial/\partial\eta^i)^2 = 0$ which implies that there is no inverse to differentiation. So it will be more difficult to define the integration. We can still define the integration by requirement of linearity and the relations

$$\int d\eta\, 1 = 0\,, \tag{5.14}$$

$$\int d\eta\, \eta = 1\,. \tag{5.15}$$

Eq. (5.14) follows from the requirement that the property of a convergent integral over anticommuting numbers

$$\int_{-\infty}^{+\infty} dx\, f(x) = \int_{-\infty}^{+\infty} dx\, f(x+a) \tag{5.16}$$

valid for any finite a, it is valid also for an integral over anticommuting numbers; one can make shifts in the integral and the measure does not change. Another way to look at (5.14) is

$$\left(\int d\eta\right)^2 = \int d\eta \int d\eta_1 = \int d\eta\, d\eta_1 = -\int d\eta_1 d\eta = -\left(\int d\eta\right)^2 \Rightarrow \int d\eta = 0\,. \qquad (5.17)$$

Eq. (5.15) can be seen as a normalization convention. The infinitesimal $d\eta_i$ are also Grassman variables:

$$\{\eta_i, d\eta_j\} = \{d\eta_i, d\eta_j\} = \{\eta_i, \eta_j\} = 0\,. \qquad (5.18)$$

Then we see that integrals and left derivatives are identical:

$$\int d\eta f(\eta) = \int d\eta (f_0 + \eta f_1) = \frac{d}{d\eta} f(\eta) = f_1\,. \qquad (5.19)$$

The integral of a derivative vanishes:

$$\int d\eta \frac{d}{d\eta} f(\eta) = \frac{d^2}{d\eta^2} f(\eta) = 0\,; \int d\eta_1 \eta_2 = -\eta_2 \int d\eta_1\,. \qquad (5.20)$$

Now let us consider the change of the integration variable:

$$\eta \to \eta' = a + b\eta\,. \qquad (5.21)$$

One gets

$$\int d\eta f(\eta) = \int d\eta' \left(\frac{d\eta}{d\eta'}\right)^{-1} f(\eta(\eta'))\,, \qquad (5.22)$$

i.e. the standard Jacobian appears inverted (sea exercise 5.1). You need this rule in order to maintain the integration property:

$$\int d\eta\eta = \int d\eta'\eta'\,. \qquad (5.23)$$

All these rules are simply generalized to the case of n real Grassman variables (see exercise 5.1).

For a complex Grassman variable η one can either work with its real and imaginary parts as two independent generators of the Grassman algebra. Equivalently, the real and imaginary parts can be replaced by η and η^* as independent generators of the Grassman algebra. Considering the Jacobian of the transformation for Grassman variables, we have

$$\int d\eta\, d\eta^* = \frac{1}{2i} \int d(\text{Re}\eta) d(\text{Im}\eta)\,. \qquad (5.24)$$

This can be generalized to n-dimensional complex Grassman variables

$$\int d\eta_1 d\eta_1^* \cdots d\eta_n d\eta_n^* = \frac{1}{(2i)^n} \int d(\text{Re}\eta_1) d(\text{Im}\eta_1) \cdots d(\text{Re}\eta_n) d(\text{Im}\eta_n)\,. \qquad (5.25)$$

Let us first consider n real Grassman variable $\eta_1 \cdots \eta_n$ with

$$\begin{cases} \{\eta_i, \eta_j\} = 0\,, \quad \left\{\frac{d}{d\eta_i}, \eta_i\right\} = 1\,, \\ \left\{\frac{d}{d\eta_i}, \frac{d}{d\eta_j}\right\} = 0\,, \\ \int d\eta_i = 0\,, \quad \int d\eta_i \eta_i = 1 \text{ (no summation on } i) \\ \int d\eta_1 d\eta_2 \eta_1 \eta_2 = -\int d\eta_1 d\eta_2 \eta_2 \eta_1 = -\int d\eta_1 \eta_1 = -1\,. \end{cases} \qquad (5.26)$$

and $\eta_i^2 = 0$, so that any function $f(\eta_1, \cdots, \eta_n)$ can be expanded in a power series in the η_i which terminates when there are at most n factors $\eta_1, \cdots, \eta_n$.

We can now consider a Gaussian integral over n Grassman variables

$$I_n = \int d\eta_1 \cdots d\eta_n \, \exp\left\{-\frac{1}{2}\eta^T A \eta\right\} , \tag{5.27}$$

where $\eta^T = (\eta_1, \cdots, \eta_n)$ and A is a real antisymmetric matrix. [3]

Now we proceed with expanding the the exponential in eq. (5.27). Each non zero term in this expansion involves an even number of factors of η_i, while non of the factors are equal. On the other hand when n is odd there is an odd number of $d\eta_i$. This means that there is at least one $\int d\eta_i = 0$ such that $I_n = 0$. When n is even the only term to be retained in the expansion of the exponential in (5.27) is the one with n factors of η_i, all the rest vanish when the integration is performed. Thus for even n

$$I_n = \int d\eta_1 \cdots d\eta_n \frac{1}{(n/2)!}\left(-\frac{1}{2}\eta^T A \eta\right)^{n/2} = (\det A)^{1/2} = \exp\left\{\frac{1}{2}\mathrm{tr}\ln A\right\} . \tag{5.28}$$

Here we do not give any demonstration for the second equality in the above relation. It is easy to see that (5.28) is correct just by checking the simple cases $n = 2$.

- n=2

$$\sum_{i,j} \eta_i A_{ij} \eta_j = \eta_1 a_{12} \eta_2 + \underbrace{\eta_1 a_{11} \eta_1}_{=0} + \eta_2 a_{21} \eta_1 + \underbrace{\eta_2 a_{22} \eta_2}_{=0} , \tag{5.29}$$

then we have

$$\begin{aligned} &\int d\eta_1 d\eta_2 \frac{1}{(2/2)!}\left[-\frac{1}{2}(a_{21}\eta_2\eta_1 + a_{12}\eta_1\eta_2)\right] \\ &= \int d\eta_1 d\eta_2 \left[-\frac{1}{2}(-a_{12}\eta_2\eta_1 - a_{12}\eta_2\eta_1)\right] = a_{12} , \end{aligned} \tag{5.30}$$

which is the same as considering

$$(\det A)^{1/2} = \left(\det\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}\right)^{1/2} = (-a_{12}a_{21})^{1/2} = a_{12} . \tag{5.31}$$

We may also use eq. (5.28) for n odd because the determinant of an antisymmetric matrix $n \times n$ with n odd is zero. One can compare (5.28) with the Gaussian integrals that we already have seen for ordinary numbers:

$$\int dz_1 \cdots dz_n \exp\left\{-\frac{z^T A z}{2}\right\} = \frac{1}{\sqrt{\det A}} . \tag{5.32}$$

[3] Note that one can start from $\eta^T M \eta$ with the most general form of M. Then, this matrix can be divided into symmetric and antisymmetric parts. One can easily see that the symmetric part of M do not contribute to the total amount of $\eta^T M \eta$. Therefore it is only the antisymmetric part of M that really matters and the symmetric part of M is redundant. Thus, the expression $\eta^T M \eta$ does not change if we add any arbitrary symmetric matrix to M. In order to remove the redundancy and assign a unique matrix to the Grassman expression $\eta^T M \eta$ we need to force M to have an specific form. This can be achieved if we restrict M to be an antisymmetric matrix. Note that this is just an arbitrary choice, which simplifies our calculations. Another possible choice to remove the redundancy would be having a strictly upper (or lower) triangular matrix for M. Here we continue with the former choice, thus we assume that M is an antisymmetric matrix denoted by A.

We see that for fermions the determinant comes at the numerator instead of the denominator.

A useful extension of eq. (5.28) is to include a linear term in the exponential:

$$\int d\eta_1 \cdots d\eta_n \exp\left\{-\frac{1}{2}\eta^T A\eta + \rho^T \eta\right\} = \sqrt{\det A}\ \exp\left\{-\frac{1}{2}\rho^T A^{-1}\rho\right\}, \tag{5.33}$$

where ρ is a Grassman variable:

$$\{\rho_i, \rho_j\} = \{\rho_i, \eta_j\} = 0\,. \tag{5.34}$$

Eq. (5.33) can be obtained form (5.28) completing the square:

$$\eta^T A\eta - 2\rho^T\eta = (\eta + A^{-1}\rho)^T A(\eta + A^{-1}\rho) + \rho^T A^{-1}\rho\,, \tag{5.35}$$

where we have used the antisymmetry of A and the fact that ρ and η are Grassman variables, and making the change of variable $\eta' = \eta + A^{-1}\rho$.

5.2 Complex Grassman integrals

In the case of complex Grassman variables the generalization of (5.28) is:

$$\int d\eta_1^* d\eta_1 \cdots \int d\eta_n^* d\eta_n e^{-(\eta^*)^T M\eta}\,, \tag{5.36}$$

Note that the matrix M does not need to be antisymmetric. [4]

We have (up to a sign):

$$\int \mathcal{D}\eta\,\mathcal{D}\eta^* e^{-(\eta^*)^T M\eta} = \det M\,, \tag{5.37}$$

where $\int \mathcal{D}\eta = \prod d\eta_i$ and $\int \mathcal{D}\eta^* = \prod d\eta_i^*$ (see exercise 5.1). The sign of these quantities depend on the order in which the factors are arranged. This formula can give us a new way to represent a determinant.

Including a linear term to the exponential, we find that

$$\int \mathcal{D}\eta\,\mathcal{D}\eta^* \exp\left\{-(\eta^*)^T M\eta + (\theta^*)^T\eta + (\eta^*)^T\theta\right\} = \det M\ \exp\left\{(\theta^*)^T M^{-1}\theta\right\}\,.$$

To obtain the above relation, as before, we can think of a change of variables as follows:

$$\begin{cases}\eta = \eta' + M^{-1}\theta\,,\\ (\eta^*)^T = (\eta'^*)^T + (\theta^*)^T M^{-1}\,,\end{cases} \tag{5.38}$$

which leads to

$$-(\eta^*)^T M\eta = -\left((\eta'^*)^T + (\theta^*)^T M^{-1}\right) M \left(\eta' + M^{-1}\theta\right)$$

[4] Similar to what we did for the real Grassman case, one can divide M into symmetric and antisymmetric parts. But, it turns out that both symmetric and antisymmetric parts of M do contribute to the total amount of $(\eta^*)^T M\eta$. (This can be easily seen by considering η_i^* and η_i as independent Grassman variables, or just by expressing the complex Grassman variables in terms of their real and imaginary parts.) Therefore M can be any general complex matrix. However, at some point in our discussion, we need to assume that M is invertible, *i.e.*, $\det(M) \neq 0$.

$$= -(\eta'^*)^T M\eta - (\theta^*)^T\eta' - (\eta'^*)^T\theta - (\theta^*)^T M^{-1}\theta\,, \tag{5.39}$$

and we finally find that

$$\begin{aligned} -(\eta^*)^T M\eta + (\theta^*)^T\eta + (\eta^*)^T\theta &= -(\eta'^*)^T M\eta + (\theta^*)^T(\eta-\eta') + (\eta^* - \eta'^*)^T\theta - (\theta^*)^T M^{-1}\theta \\ &= -(\eta'^*)^T M\eta + (\theta^*)^T M^{-1}\theta\;, \end{aligned} \tag{5.40}$$

Note that in the change of variable suggested in (5.38), one should keep in mind that a complex Grassman variable and its conjugate can be considered as two independent degrees of freedom.

Now to describe Fermi fields we make the transition to an infinite dimensional Grassman algebra with generators $C(x)$:

$$\begin{cases} \{C(x), C(y)\} = 0\,, \quad \left\{\frac{d}{dC(x)}, C(y)\right\} = \delta(x-y)\,, \\ \frac{\partial C(x)}{\partial C(y)} = \delta(x-y)\,, \quad \left\{\frac{\delta}{\delta C(x)}, \frac{\delta}{\delta C(y)}\right\} = 0\,, \\ \int dC(x) = 0\,, \quad \int dC(x)C(x) = 1\,. \end{cases} \tag{5.41}$$

We have then the following formulas:

$$\int \mathcal{D}C^*\mathcal{D}C \exp\left\{-\int dx' \int dx\, C^*(x')M(x',x)C(x)\right\} = \det M = \exp\left(\mathrm{tr}\ln M\right)\;, \tag{5.42}$$

$$\int \mathcal{D}C^*\mathcal{D}C \exp\left\{-\int dx' \int dx\, C^*(x')M(x',x)C(x) + \int dx\left[\rho^*(x)C(x) + C^*(x)\rho(x)\right]\right\}$$
$$= \det M\;\exp\left\{\int d^4x' \int d^4x \rho^*(x')M^{-1}(x',x)\rho(x)\right\}\,. \tag{5.43}$$

Depending on the choice of space (Minkowski or Euclidean), a factor of i can be introduced or removed from the matrix M.

5.3 Dirac fields and path integral quantization

We describe Dirac fields in the path integral with the Grassman variables. This makes apparent that spin fields are essentially non classical.

We take the classical Dirac fields $\psi(x)$ and $\bar\psi(x)$ as elements of an infinite dimensional Grassman algebra. The generating functional for a free fermion field described by

$$\mathcal{L}_D = \bar\psi(i\partial\!\!\!/ - m)\psi\,, \tag{5.44}$$

is

$$Z_{\rm free}[\eta,\bar\eta] = N\int \mathcal{D}\bar\psi\mathcal{D}\psi\;\exp\left\{\frac{i}{\hbar}\int d^4x\mathcal{L}_D + i\int d^4x\left[\bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)\right]\right\}\;, \tag{5.45}$$

with $N^{-1} = \langle 0|0\rangle = \int \mathcal{D}\bar\psi\mathcal{D}\psi e^{\frac{i}{\hbar}S[\psi]}$, the normalization requires $Z[0,0]=1$ and η and $\bar\eta$ are sources for the fermion fields, they are also Grassman variables, which should be treated as independent.

The Green's functions may be defined as follows:

$$
\begin{aligned}
&G^{(2n)}(x_1,\,\ldots,x_n;y_1,\,\ldots,y_n) = \frac{\langle 0|T\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\,\hat{\bar{\psi}}(y_1)\cdots\hat{\bar{\psi}}(y_n)|0\rangle}{\langle 0|0\rangle} \\
&= N\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,\psi(x_1)\cdots\psi(x_n)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)\,\exp\left\{i\int d^4x\mathcal{L}_D\right\} \\
&= \frac{1}{(i)^{2n}}\,\frac{\delta^{2n}Z_{\mathrm{free}}[\eta,\bar{\eta}]}{\delta\eta(y_n)\cdots\delta\eta(y_1)\,\delta\bar{\eta}(x_n)\cdots\delta\bar{\eta}(x_n)}\Bigg|_{\bar{\eta}=\eta=0}.
\end{aligned} \tag{5.46}
$$

We notice that $G^{(2n)}$ is antisymmetric in the index x_j and y_i:

$$
\frac{\delta^2}{\delta\eta(x_i)\delta\eta(x_j)} = -\frac{\delta^2}{\delta\eta(x_j)\delta\eta(x_i)}\,. \tag{5.47}
$$

Instead of (5.46) we can use

$$
G^{(2n)}(x_1,\,\ldots,x_n;y_1,\,\ldots,y_n) = \frac{1}{(i)^{2n}}\frac{\delta^n}{\delta\bar{\eta}(x_n)\cdots\delta\bar{\eta}(x_n)}Z_0[\eta,\bar{\eta}]\frac{\overleftarrow{\delta}^{\,n}}{\overleftarrow{\delta}\,\eta(y_n)\cdots\overleftarrow{\delta}\,\eta(y_1)}\,. \tag{5.48}
$$

It is possible to evaluate (5.43) exactly (in Minkowski space) with $M = -iB$, where $B = (i\partial\!\!\!/ - m)$:

$$
Z_{\mathrm{free}}[\eta,\bar{\eta}] = N\det[-i(i\partial\!\!\!/ - m)]\,\exp\left\{-\int d^4x\int d^4y\,\bar{\eta}(x)S_F(x-y)\eta(y)\right\} \tag{5.49}
$$

where

$$
\begin{cases}
B^{-1} = (i\partial\!\!\!/ - m)^{-1} = \frac{1}{i}S_F\,, \\
\Rightarrow\ -iB^{-1} = -S_F\,, \\
\Rightarrow\ S_F = \frac{i}{i\partial\!\!\!/ - m}\,.
\end{cases}
$$

So we can write the two point function:

$$
\begin{aligned}
G_0^{(2)}(x_1,y_1) &= \frac{1}{(i)^2}\frac{\delta^2}{\delta\eta(y_1)\delta\bar{\eta}(x_1)}Z_{\mathrm{free}}[\eta,\bar{\eta}]\Bigg|_{\bar{\eta}=\eta=0} \\
&= \frac{1}{(i)^2}\frac{\delta^2}{\delta\eta(y_1)\delta\bar{\eta}(x_1)}\exp\left\{-\int d^4x\int d^4y\bar{\eta}(x)S_F(x-y)\eta(y)\right\} \\
&= (-1)\frac{\delta}{\delta\eta(y_1)}\left\{-\int d^4yS_F(x_1-y)\eta(y)\right\}Z_{\mathrm{free}}[\eta,\bar{\eta}] = S_F(x_1-y_1)\,.
\end{aligned} \tag{5.50}
$$

So we summarize as follows

$G^{(2)}(x_1,y_1) =$ 

.

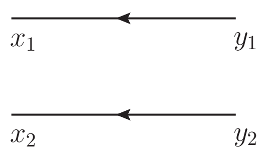
By analogy we work out the four point Green function

$$
\begin{aligned}
G_0^{(4)}(x_1,x_2;y_1,y_2) &= \frac{1}{(i)^4}\,\frac{\delta^4 Z_{\mathrm{free}}[\eta,\bar{\eta}]}{\delta\eta(y_2)\delta\eta(y_1)\,\delta\bar{\eta}(x_2)\delta\bar{\eta}(x_1)}\Bigg|_{\bar{\eta}=\eta=0} \\
&= \frac{\delta^3}{\delta\eta(y_2)\delta\eta(y_1)\,\delta\bar{\eta}(x_2)}\left\{-\int d^4yS_F(x_1-y)\eta(y)\right\}Z_{\mathrm{free}}[\eta,\bar{\eta}]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\delta^2}{\delta\eta(y_2)\delta\eta(y_1)} \left\{ - \int d^4y S_F(x_1 - y)\eta(y) \right\} \left\{ - \int d^4y S_F(x_2 - y)\eta(y) \right\} Z_{\text{free}}[\eta, \bar{\eta}] \\
&= \frac{\delta}{\delta\eta(y_2)} [-S_F(x_1 - y_1)] \left\{ - \int d^4y S_F(x_2 - y)\eta(y) \right\} Z_{\text{free}}[\eta, \bar{\eta}] \\
&- \frac{\delta}{\delta\eta(y_2)} \left\{ - \int d^4y S_F(x_1 - y)\eta(y) \right\} [-S_F(x_2 - y_1)] Z_{\text{free}}[\eta, \bar{\eta}] \\
&= S_F(x_1 - y_1) S_F(x_2 - y_2) - S_F(x_1 - y_2) S_F(x_2 - y_1) \, .
\end{aligned} \tag{5.51}
$$

So, as for the two-point Green's function, we summarize as follows

$G^{(4)}(x_1, x_2; y_1, y_2) =$



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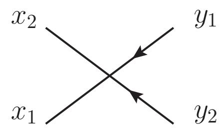


.

Let us stress some differences with respect to the real scalar field:

- the propagator has an arrow. Dirac particles are not their antiparticles (antiparticles propagate in the inverse sense of the arrow),
- the minus sign is related to the fermion character.

5.4 Path Integral description of the interacting fermions

In order to consider interacting fermions we have to consider the full Lagrangian density: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$. The interacting term describes the interaction between fermions and other fields. For example, for the Yukawa theory we have

$$
\mathcal{L}_I = g\bar{\psi}\gamma^5\psi\phi \, . \tag{5.52}
$$

We can also consider QED, where the interacting Lagrangian is

$$
\mathcal{L}_I = e\bar{\psi}\gamma_\mu A^\mu \psi \, . \tag{5.53}
$$

Accordingly the generating functional accounting for interactions reads

$$
Z[\eta, \bar{\eta}, ...] = \exp\left\{ i \int d^4x \, \mathcal{L}_I \left(\frac{\delta}{i\delta\eta(x)}, \frac{\delta}{i\delta\bar{\eta}(x)}, ... \right) \right\} Z_{\text{free}}[\eta, \bar{\eta}, ...] \, , \tag{5.54}
$$

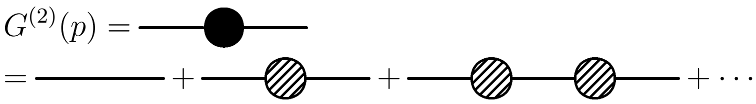
where we allow for more Grassmann variables as denoted with the dots in the argument of the generating functional.

6 UV divergences and renormalization in field theory

6.1 Introduction

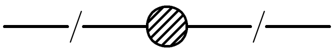
We have already noticed in some concrete examples that the initial parameters of the classical theory can not stay the same when quantum corrections are introduced.
For example we have seen that

$G^{(2)}(p) =$

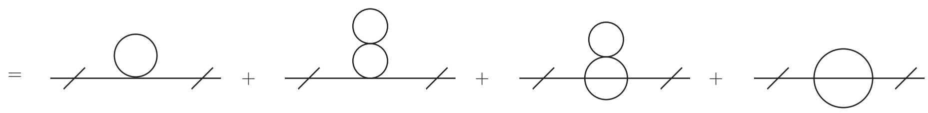


$$= \frac{i}{p^2 - m^2 - \Sigma(p)} = \frac{i}{p^2 - m_R^2} \tag{6.1}$$

where m_R is a renormalized or physical mass: $m_R^2 = m^2 + \Sigma(p)$, and $\Sigma(p) =$  is the self energy given by the sum of 1PI amputated diagrams (up to order λ^2) :



$= -i\Sigma(p) =$



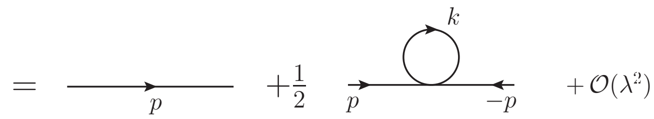
$$. \tag{6.2}$$

We have therefore

$$\Gamma^{(2)}(p) = G^{(2)}(p) = i \;\Rightarrow\; \Gamma^{(2)}(p) = p^2 - m^2 - \Sigma(p)\,. \tag{6.3}$$

In particular at order λ we calculated

$$G^{(2)}(p) = \frac{i}{p^2 - m^2 - \frac{\lambda}{2}D_F(0)}$$



$$\tag{6.4}$$

with

$$D_F(0) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\,, \tag{6.5}$$

and in the same way we calculated at order λ the classical field equations

$$\left(\partial_x^2 + m^2 + \frac{\lambda}{2}D_F(0)\right)\phi_{\rm cl}(x) = -\frac{1}{6}\lambda\phi_{\rm cl}^3(x) + \mathcal{O}(\lambda^2)\,, \tag{6.6}$$

so that at this order

$$m_R^2 = m^2 + \frac{\lambda}{2}D_F(0) + \mathcal{O}(\lambda^2)\,, \tag{6.7}$$

where $\lambda D_F(0)/2$ is a quantum correction that appears when the field interacts. Then it is apparent that the interactions have the effect to shift (="renormalize") the squared mass from its initial value m^2, valid when there are no interactions. The parameter m^2 that appears in the Lagrangian is the physical mass *only in the classical limit* (i.e.

no quantum corrections, $\hbar = 0$). The same is true for the coupling constant λ. This would have become apparent if we had calculated the order λ^2, indeed at this order the interactions generate quantum corrections that shift the coupling constant from its "bare" value λ to a renormalized λ_R.

Now the only quantities that we can measure are the renormalized quantities since we can not turn the interaction on and off as we want. For example we my be able to measure the $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$ for a particular choice of the momenta p_i and define this to be the renormalized coupling constant λ_R. Then we could calculate $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$ or any other proper Green's function $\tilde{\Gamma}^{(n)}$ using the Feynman rules for arbitrary momenta as a function of the renormalized λ_R and m_R^2 (that are measured input). We would expect that any perturbative approximation we make would be more successful in predicting $\tilde{\Gamma}^{(n)}$ at e.g. GeV values for the momenta, if we use renormalized parameters defined at the GeV momenta, rather than those defined at eV or TeV momentum scale.

This idea that we may vary the energy scale at which we choose to define our renormalization parameters λ_R and m_R^2 is the key of the renormalization group equations (RGE).

The point of this discussion is that the Lagrangian we have been using up to now is actually given in terms of the "bare" or "unrenormalized" quantities, that we can denote with a subscript "B":

$$\phi_B,\ m_B^2,\ \lambda_B\,. \tag{6.8}$$

Our aim is to calculate physical (=observable) quantities (like cross sections) as function of the renormalized quantities ϕ_R, m_R^2, λ_R (or simply ϕ, m^2, λ). Then the Lagrangian of the scalar field theory we have used up to know can be written as follows:

$$\mathcal{L}_B = \frac{1}{2}(\partial^\mu \phi_B)(\partial_\mu \phi_B) - \frac{1}{2} m_B^2 \phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4\,, \tag{6.9}$$

and the corresponding Green's functions calculated from this $\mathcal{L}_B$ are the "bare" ones: $\Gamma_B^{(n)}$, $\Gamma_B^{(n)}$, ...

Also the choice of the factor 1/2 in front of the derivative term amounts to an arbitrary choice of the field strength normalization and we will have to introduce a renormalized field ϕ_R (or ϕ) differing from ϕ_B by an overall multiplicative constant. We will see this in details in a moment.

Now let me stress that what I have discussed about renormalization up to now is pretty general and it is not related in particular to the emergence of infinities $\Rightarrow$ even in a totally finite theory we would still have to renormalize physical quantities. Let us make a concrete example. Let me consider an electron moving inside a solid. Due to the interaction of the electron with the lattice of the solid the effective mass, m^*, of the electron (giving its response to the externally applied force) *is different* from the mass of the electron measured outside the solid.
$\Rightarrow$ the electron mass is changed from m to m^* by the interaction of the electron with the lattice.
In this simple case both m and m^* can be measured placing the electron inside or outside the solid. In this case clearly $m - m^*$ is finite since both m and m^* are finite and measurable.

For the relativistic field theory the situation is the same apart from two important differences:

1) the renormalization due to the interactions is in general *infinite*, corresponding to divergent loop diagrams,

2) there is no way to switch off the interactions, therefore the quantities in absence of interactions (unrenormalized or bare) are not measurable.

For example in QED the difference between the m_B and m_R for the electron is infinite and m_B cannot be measured because the electron interacts constantly with the virtual photon field and there is no way to switch off this interaction.

Since the shifts in QFT are infinite and the renormalized parameters are finite, it follows that the bare quantities are infinite.

Then the use of renormalized quantity is necessary if we want to avoid infinities in the calculations of physical quantities. This raises the question of whether *all infinities* may be eliminated from the theory by the use of m_R^2, λ_R, ϕ_R. Indeed one can show that for the theory in (6.9) the renormalization of mass, coupling constant and the field is sufficient to make finite any physical observable $\Rightarrow$ all Green's functions of the theory are finite.

A QFT is called renormalizable if it is made finite by the renormalization of only the parameters and fields appearing in $\mathcal{L}_B$. The theories of strong and electroweak forces are renormalizable. At the moment we do not have a renormalizable QFT to describe gravitation.

The renormalization program has been originally formulated for QED by Feynman (1948), Schwinger (1948, 1949), Tomonaga (1948) and Dyson (1949) $\rightarrow$ this program has been very successful as the agreement between experiment and theory for the standard model is spectacular.

Let us discuss the type of infinities that show up in the $\lambda\phi^4$ theory. The infinity associated with the mass at order λ is given by the following expression

$$\frac{\lambda_B}{2} D_F(0) = \frac{\lambda_B}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_B^2 + i\epsilon} \,. \tag{6.10}$$

If we want to evaluate (6.10), since there are no other singularities than the poles shown in figure 6.1, we can perform a Wick rotation of the contour in anticlockwise direction to lie along the imaginary p_0 axis form $-i\infty$ to $+i\infty$, so that

$$p_0 = -ip_4 \ ,$$

and the integration over p_4 runs from $-\infty$ to $+\infty$. So we have, $p_E^2 = k_4^2 + \mathbf{p}^2$ and

$$\frac{\lambda_B}{2} D_F(0) = (-i)(i)\lambda_B \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{-p_E^2 - m_B^2} \,,$$

and because $d^4p_E \sim |p|^3 d|p|$ the integral is UV divergent as Λ^2, that we call a "quadratically divergent" integral.

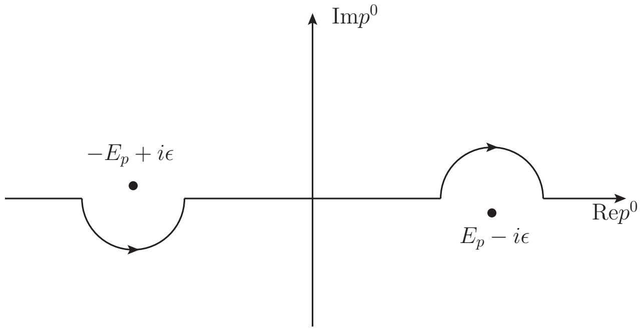
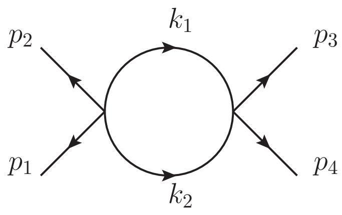


Figure 6.1: Poles of the propagator in the loop integral.

In the same way considering a contribution of order λ_B^2 to $\tilde{\Gamma}_B^{(4)}$ we have:



and the diagram corresponds to the loop integral

$$\frac{\lambda_B^2}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{1}{k_1^2 - m_B^2 + i\epsilon}\frac{1}{(p_1+p_2+k_1)^2 - m_B^2 + i\epsilon}\,, \tag{6.11}$$

where we obtain from the momentum conservation $-k_2 = p_1 + p_2 + k_1$. This is also a divergent quantity. We can make an analytical continuation to Euclidean space. For large (Euclidean) k_1 each propagator gives $1/k_1^2$ and the integration measure is $d^4k_1 \sim d|k||k_1|^3$, so we see that the integral diverges logarithmically $\sim \ln\Lambda$.

From these examples we see how we can generalize the argument to identify which Green's functions are divergent. The naive superficial "degree of divergence" D of a diagram is given by (for $\lambda\phi^4$ theory):

$$D = 4L - 2I\,, \tag{6.12}$$

where L is the number of independent loop momenta and I is the number of internal lines. This is because each loop had a d^4k and each scalar internal line has a behaviour for large k as $1/k^2$. Then we have

- $D = 0$: logarithmically divergent,
- $D = 1$: linearly divergent,
- $D = 2$: quadratically divergent.

We also know that $L < I$ due to momentum conservation at each vertex. For a connected diagram we have

$$L = I - V + 1\,, \tag{6.13}$$

where V is the number of vertices (only $V-1$ because of the overall momentum conservation). Substituting (6.13) into (6.12) we get:

$$D = 2I - 4V + 4\,. \tag{6.14}$$

On the other hand we have the relation

$$4V - 2I = E\,, \tag{6.15}$$

where E is the number of the external lines, since each vertex has four lines and each internal line remove two of those. Then we have finally:

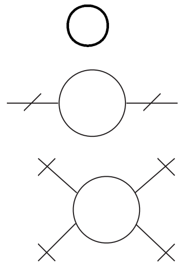
$$D = 4 - E\,, \tag{6.16}$$

and we see that *the degree of divergence is independent on how many vertices are in the diagram.* Since in $\lambda\phi^4$ E has to be even (since the theory is invariant under $\phi \to -\phi$ all amplitudes with an odd number of external legs vanish), we have only 3 proper Green's functions that are superficially divergent:

$\tilde\Gamma^{(0)}$:

$\tilde\Gamma^{(2)}$:

$\tilde\Gamma^{(4)}$:



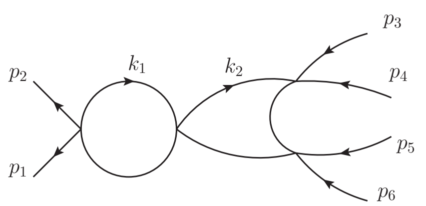
$\tilde\Gamma^{(0)}$ can be simply ignored being the unobservable vacuum energy shift. The other two amplitudes contain three infinite constants $\to$ our goal is to reabsorb these three constants in three unobservable parameters of the theory: m_B^2, λ_B and ϕ_B.

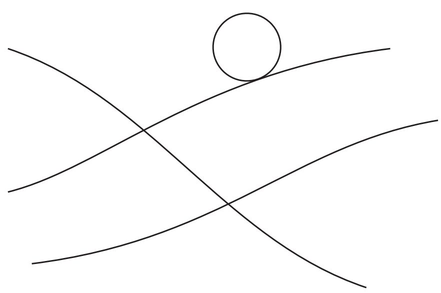
However, are these really all divergences? When a graph γ is convergent?

- if $D(\gamma) < 0$ when it has a single loop
- if $D(\gamma_{tot}) < 0$ and $D(\gamma_i) < 0$ for each subgraph (Weinberg theorem), in the case of many loops.

For $E > 4 \Rightarrow D(\gamma_{tot}) < 0$ so that a diagram with more than 4 external lines can be divergent only if contains some divergent subgraph γ_i such that $D(\gamma_i) \leq 4$. Indeed there are many contributions to Green's functions with more than four external lines that are divergent (that is why we called D the superficial degree of divergence).

Let us consider for example the following two loop contributions to $\tilde\Gamma^{(6)}$:

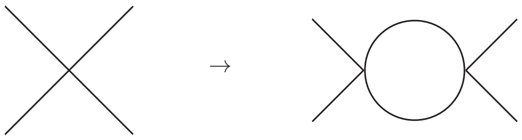




although $E = 6$, they are divergent. The integration over the loop momentum k_1 diverges. We notice that this divergence in k_1, while k_2 is kept fixed, is the divergence already encountered in $\tilde{\Gamma}^{(4)}$:

$$\sim \frac{\lambda_B^2}{2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{k_1^2 - m_B^2 + i\epsilon} \frac{1}{(p_1 + p_2 + k_1)^2 - m_B^2 + i\epsilon} ,$$

with $D = 0$, logarithmically divergent. Clearly there will be a divergent contribution to each Green's function whenever in a particular diagram we make the replacement:



. (6.17)

We note that $\tilde{\Gamma}^{(2)}$ and $\tilde{\Gamma}^{(4)}$ are *primitive divergences* of $\lambda\phi^4$.

Weinberg theorem tells us when a diagram is convergent. But, these conditions are sufficient for the convergence of a diagram but not always necessary. It may happen that some invariance/symmetry of a theory makes a diagram more convergent than the degree of divergence coming from $D(\gamma)$. This does not happen in $\lambda\phi^4$ but it happens in QED for *gauge invariance and charge conjugation invariance.*

If the theory has to be renormalizable the only divergences which arise are those which arise from mass, coupling constant and field renormalization.

UV behaviour of QFT

Three possible types of UV behaviour occur in QFTs:

- *Super-renormalizable theory*: only a finite number of Feynman diagrams superficially diverge,
- *Renormalizable theory*: only a finite number of amplitudes (primitive divergences) superficially diverge; however divergences appear at all orders of perturbation theory,
- *Non-renormalizable theory*: all amplitudes are divergent at a sufficiently high order in perturbation theory.

The UV behaviour of QFT depends on the number of dimensions we work with. Indeed in $\lambda\phi^4$: $D = 4L - 2I$, where $d = 4$ is the number of dimensions.

For example, let us consider a $\lambda\phi^n$ theory in a generic d:

$$\mathcal{L}_B = \frac{1}{2}(\partial^\mu \phi_B)(\partial_\mu \phi_B) - \frac{1}{2} m_B^2 \phi_B^2 - \frac{\lambda_B}{n!}\phi_B^n . \tag{6.18}$$

Then we consider

$$D = dL - 2I$$

and by using

$$\begin{cases} L = I - V + 1 \\ nV = E + 2I \Rightarrow I = \frac{nV-E}{2} \end{cases} \tag{6.19}$$

we obtain

$$\begin{aligned} D &= dL - 2I \\ &= d - dV + dI - 2I \\ &= d - dV + I(d-2) \\ \Rightarrow \quad D &= d - dV + (d-2)\frac{nV-E}{2} = d + V\left[n\frac{(d-2)}{2} - d\right] - E\left(\frac{d-2}{2}\right) \end{aligned} \tag{6.20}$$

We observe that in $d = 4$, $\lambda\phi^4$ is renormalizable because

- D no longer depends on V, indeed $n(d-2)/2 - d = 0$,
- $D = 4 - E$,

while for higher powers of ϕ^n, $n > 4$, the theory is not renormalizable, because D increases with the number of vertices and all Green functions are divergent of higher order in perturbation theory.
In $d = 3$ we have that $\lambda\phi^6$ is renormalizable, $6(3-2)/2-3 = 0$. In $d = 3$ we also have that $\lambda\phi^4$ is super-renormalizable, $4(3-2)/2-4 = -2$. In $d = 2$ any ϕ^n is super-renormalizable: $n(d-2)/2 - d = -2$ and

$$D = 2 - 2V\,, \tag{6.21}$$

and thus D decreases with the number if vertices.

The expression in (6.20) can be obtained also via dimensional analysis. We start with noticing that, when $\hbar = 1$, the action, $S = \int d^4x\mathcal{L}$ is dimensionless. Then

$$\left[d^dx\right] = m^{-d}\,, \quad [\mathcal{L}] = m^d\,,$$

where with m we mean the dimension of an energy. Then from $\mathcal{L}$ we can obtain the dimension for the field and the coupling for any d, $[\phi]$ and $[\lambda]$ respectively. From $\mathcal{L}$ in (6.18) we have:

$$[\partial^\mu\phi\partial_\mu\phi] = m^d \Rightarrow \left[\phi^2\right] = m^{d-2} \Rightarrow [\phi] = m^{\frac{d-2}{2}}\,, \tag{6.22}$$

$$[\lambda\phi^n] = m^d \Rightarrow [\lambda]\, m^{n\frac{d-2}{2}} = m^d \Rightarrow [\lambda] = m^{d-n\frac{d-2}{2}}\,. \tag{6.23}$$

Now let us consider a diagram with E external lines . One could see it as arising from an interaction term $\eta\phi^E$ in $\mathcal{L}$, then

$$\left[\eta\phi^E\right] = m^d \Rightarrow [\eta] = m^{d-E\frac{(d-2)}{2}}\,,$$

so any amputated diagram with E external lines has dimension $m^{d-E\frac{(d-2)}{2}}$. In our $\lambda\phi^n$ theory, if the diagram has V vertices the divergent part is $\lambda^V\Lambda^D$, and applying dimensional analysis we write

$$d - E\frac{(d-2)}{2} = V\left[d - n\frac{d-2}{2}\right] + D\,, \tag{6.24}$$

which reproduce eq. (6.20). However in eq. (6.20) the quantity multiplying V is minus the dimension of the coupling constant λ, so we can say (recalling $[\lambda] = m^{d-n(d-2)/2}$):

- *Super-renormalizable theory*: the coupling constant has a positive mass dimension;
- *Renormalizable theory*: the coupling constant is dimensionless;
- *Non-renormalizable theory*: the coupling constant has negative mass dimension.

Also intuitively: consider contributions to cross sections in higher orders in $\sim \lambda^n$ and if $[\lambda] < 0$ one will have to compensate this negative mass dimension with positive powers of the cut off scale Λ resulting to divergent contributions in the limit $\Lambda \to \infty$.

6.2 Field strength renormalization

We are here studying singularities of the Feynman diagrams viewed as a function of the external momenta, or the analytic structure of Green functions. Let us consider the two-point Green function

$$G^{(2)} = \langle \Omega | \mathrm{T}\, \phi(x)\phi(y) | \Omega \rangle \quad \text{or} \quad \langle \Omega | \mathrm{T}\, \psi(x)\bar{\phi}(y) | \Omega \rangle \,. \tag{6.25}$$

In a free theory $G^{(2)} = \langle 0 | \mathrm{T}\, \phi(x)\phi(y) | 0 \rangle$ is the amplitude for a particle to propagate from y to x. *Is this interpretation still valid in an interacting theory*?

We will make here a general analysis based only on special relativity and quantum mechanics and not dependent on the nature of the interaction or on the expansion in perturbation theory.[5] For the actual calculation however we take a scalar field.

We insert in $\langle \Omega | \mathrm{T}\, \phi(x)\phi(y) | \Omega \rangle$ the identity operator, namely a complete set of states, eigenstates of the fully interacting Hamiltonian $\hat{H}$ that here we label just with H. Given $\mathbf{P}$ the momentum operator, we have $[\mathbf{P}, H] = 0$ and we can choose the the eigenstate of the Hamiltonian to be also the eigenstate of $\mathbf{P}$. We label them with $|\lambda_{\mathbf{p}}\rangle$ and we have

$$H|\lambda_{\mathbf{p}}\rangle = E_{\mathbf{p}}(\lambda)|\lambda_{\mathbf{p}}\rangle \,, \quad \mathbf{P}|\lambda_{\mathbf{p}}\rangle = \mathbf{p}|\lambda_{\mathbf{p}}\rangle \,,$$

each $|\lambda_{\mathbf{p}}\rangle$ is related by a Lorentz boost to the corresponding state at rest, called $|\lambda_0\rangle$, which obeys $\mathbf{P}|\lambda_0\rangle = 0$. We can have these types of $|\lambda_{\mathbf{p}}\rangle$:

- 1-particle state $|1_{\mathbf{p}}\rangle$ with $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ $(m \neq m_0)$,
- bound states with no analogy in the free theory,
- 2, ...,N particle states formed by 1-particle states and bound states.

All those states are created from the interacting vacuum $|\Omega\rangle$. However the crucial difference compared to the free theory is that ϕ cannot be simply written as a superposition of its Fourier amplitudes $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ because the field is not free. This means that acting with ϕ over $|\Omega\rangle$ does not produce 1-particle states as in the free theory. Therefore, we have to account for a field strength renormalization, Z.

[5] For definition of Fock space see notes of "*Relativity, Particles and Fields*."

If $H|\lambda_0\rangle = E_0|\lambda_0\rangle$ and $\mathbf{P}|\lambda_0\rangle = 0$, by Lorentz invariance all boosts of $|\lambda_0\rangle$ are also eigenstates of H and have all possible three-momenta. In the same way any eigenvalue of H with some definite momentum can be written as the boost of some $|\lambda_0\rangle$. The eigenstates of $P^\mu = (H, \mathbf{P})$ organize themselves in hyperboloids.

6.2.1 Kallen-Lehmann representation

Our scope is to compute the general form of the interacting Feynman propagator

$$\langle\Omega|\mathrm{T}\phi(x)\phi(y)|\Omega\rangle\,, \tag{6.26}$$

and establish its physical interpretation. We can not rely any more on the free field Fourier decomposition but we will use:

- Lorentz invariance,
- completeness relations and vacuum invariance,
- relation between $|\lambda_0\rangle$ and $|\lambda_{\mathbf{p}}\rangle$ via Lorentz boost.

Our theory is invariant under Poincaré group: $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$, then a unitary transformation $U(\Lambda, a)$ of this group exists in the Hilbert space of the theory such that:[6]

$$U(\Lambda, a)\phi(x)U^\dagger(\Lambda, a) = \phi(\Lambda x + a)\,. \tag{6.27}$$

In particular for translations we have

$$U(a) = e^{ia^\mu P_\mu}\,, \tag{6.28}$$

and

$$\phi(x) = e^{ix^\mu P_\mu}\phi(0)e^{-ix^\mu P_\mu}\,, \tag{6.29}$$

and P^μ is the four-momentum of the field, it can be used to classify the states. At least for sufficiently small coupling constant, we can expect the spectrum of P^μ to have the same characteristics as in the free field case. We assume there exists a non degenerate ground state $|\Omega\rangle$, *i.e.*, the vacuum of the interacting theory. The vacuum must be invariant under $U(\Lambda, a)$, so we have

$$U(a)|\Omega\rangle = |\Omega\rangle\,, \quad U(\Lambda)|\Omega\rangle = |\Omega\rangle\,, \quad P_\mu|\Omega\rangle = 0\,, \quad \Lambda\,|\Omega\rangle = 0\,. \tag{6.30}$$

Then we also have the 1-particle states

$$P_\mu|p\rangle = p_\mu|p\rangle\,, \quad P^2|p\rangle = p^2|p\rangle = m^2|p\rangle\,, \tag{6.31}$$

where m is the physical mass of the particle, and many particle states

$$P_\mu|p,\alpha\rangle = p_\mu|p,\alpha\rangle\,, \quad P^2|p,\alpha\rangle = p^2|p,\alpha\rangle\,, \tag{6.32}$$

[6] See Appendix C of notes of "*Relativity, Particles and Fields.*"

where α are degeneration indices. The spectrum of P^2 consist of two isolated point at 0 and m^2 and the real semi-axes from $4m^2$ to $+\infty$ (there may be bound states at $4m^2 - E_b$).

We can adopt these normalization for the states

$$\begin{cases} \langle 0|0\rangle = 1, \quad \langle p'|p\rangle = 2p_0\delta^3(\mathbf{p}-\mathbf{p}'), \\ \langle p,\alpha|p',\alpha'\rangle = \delta^4(p-p')\delta_{\alpha,\alpha'}, \end{cases}$$

and we have

$$\langle 0|p,\alpha\rangle = \langle 0|p\rangle = \langle p'|p,\alpha\rangle = 0. \tag{6.33}$$

Moreover

$$U(\Lambda)|p\rangle = |\Lambda p\rangle, \quad U(\Lambda)|p,\alpha\rangle = |\Lambda p,\alpha\rangle. \tag{6.34}$$

In the free case we have $\langle 0|\phi(x)|p\rangle = e^{-ip\cdot x}$, which is a scalar function. We use the conventions

$$\begin{cases} [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \\ \phi_0(x) = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}). \end{cases} \tag{6.35}$$

In the interacting case we do not know how to calculate it, but we can use completeness relations. The completeness relation can be written as:

$$|\Omega\rangle\langle\Omega| + \int \frac{d^3p}{2p_0}|p\rangle\langle p| + \sum_\alpha \int d^4k|p,\alpha\rangle\langle p,\alpha| = \mathbb{1}, \tag{6.36}$$

due to Poincaré invariance

$$\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|e^{ix^\mu P_\mu}\phi(0)e^{-ix^\mu P_\mu}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle = 0, \tag{6.37}$$

in $\lambda\phi^4$ (odd Green's functions vanish in $\lambda\phi^4$). We also can write

$$\begin{cases} \langle\Omega|\phi(x)|p\rangle = \langle\Omega|\phi(0)|p\rangle\, e^{-ip\cdot x}, \\ \\ \langle\Omega|\phi(x)|p,\alpha\rangle = \langle\Omega|\phi(0)|p,\alpha\rangle\, e^{-ip\cdot x}. \end{cases} \tag{6.38}$$

Then we can insert the identity in (6.36) in a correlator

$$\begin{aligned} \langle\Omega|\phi(x_1)\mathbb{1}\phi(x_2)|\Omega\rangle &= \int \frac{d^3p}{2p_0}\langle\Omega|\phi(0)|p\rangle\langle p|\phi(0)|\Omega\rangle e^{-ip\cdot(x_1-x_2)} \\ &+ \sum_\alpha \int d^4k\langle\Omega|\phi(0)|p,\alpha\rangle\langle p,\alpha|\phi(0)|\Omega\rangle e^{-ip\cdot(x_1-x_2)}, \end{aligned} \tag{6.39}$$

where the contribution of $|\Omega\rangle\langle\Omega|$ in (6.36) is zero due to (6.37). Then due to Lorentz invariance of the vacuum

$$\begin{aligned} &\sum_\alpha \langle\Omega|\phi(0)|p,\alpha\rangle\langle p,\alpha|\phi(0)|\Omega\rangle \\ &= \sum_\alpha \langle\Omega|U^\dagger(\Lambda)U(\Lambda)\phi(0)U^\dagger(\Lambda)U(\Lambda)|p,\alpha\rangle\, \langle p,\alpha|U^\dagger(\Lambda)U(\Lambda)\phi(0)U^\dagger(\Lambda)U(\Lambda)|\Omega\rangle \\ &= \sum_\alpha \langle\Omega|\phi(0)|\Lambda p,\alpha\rangle\langle\Lambda p,\alpha|\phi(0)|\Omega\rangle = \frac{\theta(p_0)}{(2\pi)^3}\rho_c(p^2). \end{aligned} \tag{6.40}$$

The second equality in (6.40) is obtained using $U(\Lambda)\phi(0)U^\dagger(\Lambda) = \phi(\Lambda \cdot 0) = \phi(0)$. The third equality says that the whole expression is only a function of p^2 and it vanishes if $p^0 < 0$. Recall that we can always operate a Lorentz transformation (rotation or boost), in particular a boost can convert any state to a state with $\mathbf{p} = 0$. Since the sum is Lorentz invariant, one can conclude that it is a scalar function of p^2. Moreover we introduced $\theta(p_0)$ due to the restrictions on the field energy and momentum that $p^2 \geq 0$ and $p_0 \geq 0$. One can also show that $\rho_c(p^2)$ *is a real and positive function*, because the left side of (6.40) is real and positive. In the same way we obtain:

$$\langle 0|\phi(0)|p\rangle\langle p|\phi(0)|0\rangle = \frac{Z}{(2\pi)^3}\,, \tag{6.41}$$

where Z is a real positive constant ($p^2 = m^2$ for one particle state). Then using (6.40) and (6.41) we obtain

$$\begin{aligned}
\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle &= Z\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}e^{-ip\cdot(x_1-x_2)} + \int\frac{d^4p}{(2\pi)^3}\theta(p_0)\rho_c(p^2)e^{-ip\cdot(x_1-x_2)}\\
&= \int\frac{d^4p}{(2\pi)^3}\theta(p_0)\left[Z\delta(p^2-m^2)+\rho_c(p^2)\right]e^{-ip\cdot(x_1-x_2)}\\
&= \int d\sigma^2\rho(\sigma^2)\int\frac{d^4p}{(2\pi)^3}\theta(p_0)\delta(p^2-\sigma^2)e^{-ip\cdot(x_1-x_2)}\,,
\end{aligned} \tag{6.42}$$

with

$$\rho(\sigma^2) = Z\delta(\sigma^2-m^2)+\rho_c(\sigma^2)\,. \tag{6.43}$$

In order to get the the result in eq. (6.42) we made use of

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0} = \int\frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\bigg|_{p_0>0}\,. \tag{6.44}$$

Now we are going to use (6.44) again to write (6.42) in a useful way. Considering

$$\int\frac{d^4p}{(2\pi)^3}\delta(p^2-\sigma^2)e^{-ip\cdot(x_1-x_2)}\bigg|_{p_0>0} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}^2+\sigma^2}}e^{-i\left(\sqrt{\mathbf{p}^2+\sigma^2}\,(x_1^0-x_2^0)\,-\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)\right)}\,, \tag{6.45}$$

(6.42) reads

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \int d\sigma^2\rho(\sigma^2)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}^2+\sigma^2}}e^{-i\left(\sqrt{\mathbf{p}^2+\sigma^2}\,(x_1^0-x_2^0)\,-\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)\right)}\,. \tag{6.46}$$

We need also to use the spectral representation of the θ function, which can be written as

$$\theta(x)\,e^{-i\lambda' x} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}d\lambda\frac{e^{i\lambda x}}{\lambda+\lambda'-i\epsilon} = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty}d\lambda\frac{e^{-i\lambda x}}{\lambda-\lambda'+i\epsilon}\,, \tag{6.47}$$

where λ' is a real number. Now we are at a point that we can put all ingredients together to calculate the two-point Green functions of the interacting theory as

$$\begin{aligned}
\langle\Omega|T\,\phi(x_1)\phi(x_2)|\Omega\rangle &= \theta(x_1^0-x_2^0)\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle + \theta(x_2^0-x_1^0)\langle\Omega|\phi(x_2)\phi(x_1)|\Omega\rangle\\
&= \int d\sigma^2\rho(\sigma^2)\frac{(-1)}{2\pi i}\int\frac{d^4p}{(2\pi)^3}\frac{e^{-ip\cdot(x_1-x_2)}}{2\sqrt{\mathbf{p}^2+\sigma^2}}\left[\frac{1}{p_0-\sqrt{\mathbf{p}^2+\sigma^2}+i\epsilon}+\frac{1}{-p_0-\sqrt{\mathbf{p}^2+\sigma^2}+i\epsilon}\right]
\end{aligned}$$

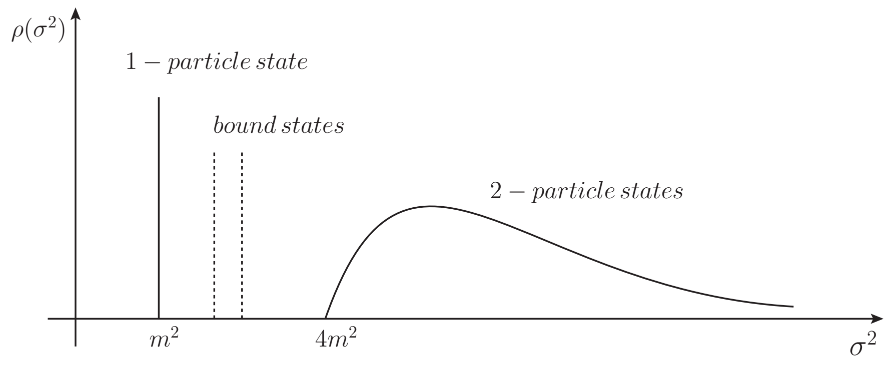


Figure 6.2: The spectral function $\rho(\sigma^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

$$
\begin{aligned}
&= i\int d\sigma^2 \rho(\sigma^2) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x_1-x_2)}}{(p_0 - \sqrt{\mathbf{p}^2+\sigma^2}+i\epsilon)(p_0+\sqrt{\mathbf{p}^2+\sigma^2}-i\epsilon)} \\
&= \int d\sigma^2 \rho(\sigma^2) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-\sigma^2+i\epsilon} e^{-ip\cdot(x_1-x_2)} \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)} \left[\frac{iZ}{p^2-m^2+i\epsilon} + \int d\sigma^2 \rho_c(\sigma^2) \frac{i}{p^2-\sigma^2+i\epsilon} \right]
\end{aligned} \tag{6.48}
$$

where we have used the definition in (6.43), and we have obtained the so called Källen-Lehmann representation:

$$
G^{(2)}(p^2) = \frac{iZ}{p^2-m^2+i\epsilon} + \int_{4m^2} d\sigma^2 \rho_c(\sigma^2) \frac{i}{p^2-\sigma^2+i\epsilon}\,. \tag{6.49}
$$

We note that $\rho_c(\sigma^2)$ is non vanishing only for $\sigma^2 \geq 4m^2$, so the integral in (6.49) is a regular function in $p^2 = m^2 \Rightarrow G^{(2)}(p^2)$ has only a pole corresponding to the physical mass. *The spectral function $\rho_c(p^2)$ gives the total probability for $\phi(0)$ to produce a multiparticle state of momentum p from the vacuum.* Let us now look at the figure 6.2 that shows the spectral function $\rho_c(\sigma^2)$ for a typical interacting field theory. The one particle states contribute with a delta function at m^2. Multiparticle states have a continuous spectrum beginning at $4m^2$. There may also be bound states.

The field strength renormalization can be written as follows

$$
\frac{Z}{(2\pi)^3} = \langle 0|\phi(0)|p\rangle\langle p|\phi(0)|0\rangle = \langle 0|\phi(0)|\Lambda p\rangle\langle \Lambda p|\phi(0)|0\rangle = \langle 0|\phi(0)|m\rangle\langle m|\phi(0)|0\rangle\,. \tag{6.50}
$$

In the last step the boost has been chosen such that $\mathbf{p} = 0$ and then m is the exact mass of a single particle $\rightarrow$ the exact energy eigenvalue at rest. It is the physical, renormalized mass.

So we have the following analytic structure of $G^{(2)}(p^2)$ in the complex p^2-plane for a typical interacting theory in figure 6.3. We see that one particle states give an isolated pole, while states with two or more free particles give a brunch cut, bound states give additional poles (in fact they would appear as additional delta functions in $\rho_c(\sigma^2)$).

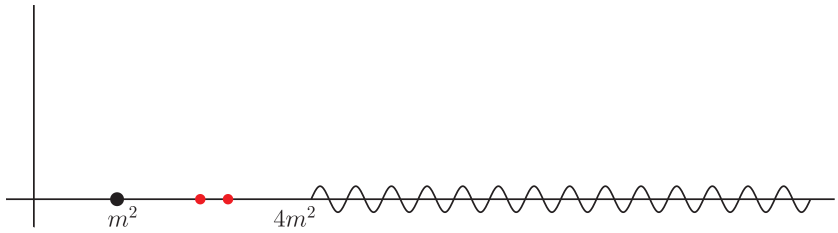


Figure 6.3: Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut (wiggled line), while bound states give additional poles (red dots).

We can compare $G^{(2)}(p^2)$ in (6.49) with the two-point correlation function in a theory of a free scalar field:

$$G^{(2)}(p^2) = \frac{i}{p^2 - m^2 + i\epsilon} = \int d^4x e^{-ik\cdot x} \langle 0|T\,\phi(x)\phi(0)|0\rangle\,, \tag{6.51}$$

for $x_0 > 0$ this gives the amplitude for a free particle to propagate from 0 to x. Our eq. (6.49) shows that the two-point function take a similar form in the theory on an interacting scalar field. The general expression is *a sum of scalar propagation amplitudes for states created from the vacuum by the field operator* $\phi(0)$, with

- the field strength renormalization factor $Z/(2\pi)^3 = |\langle p|\phi(0)|0\rangle|^2$ is the probability to create a state of momentum p from the vacuum. In the free theory $|\langle p|\phi(0)|0\rangle|^2 = 1/(2\pi)^3$,
- moreover (6.49) contains contributions from multiparticle states with a continuous spectrum $\Rightarrow$ *in the free theory* $\phi(0)$ *can create only a single particle from the spectrum.*

It is important in the field theory treatment that the contributions from one-particle and multiparticle intermediate states can be distinguished by the strength of their analytic singularities. These statements generalize to higher-point correlation functions.
Our analysis generalizes directly to the two-point functions of higher-spins fields. In such case the manipulations related to the Lorentz transformations are more complicated since now fields have a non-trivial transformation law under the boosts (spin matrix). The two point function will have the structure:

$$G^{(2)}(p^2)\frac{iZ_2(\not{p} + m)}{p^2 - m^2 + i\epsilon} + \cdots\,. \tag{6.52}$$

Here the dots stand for the brunch cut part and Z_2 is the probability for $\psi(0)$ to create or annihilate an exact one particle states:

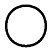
$$\langle 0|\psi(0)|p,s\rangle = \sqrt{Z_2}u^s(p)\,. \tag{6.53}$$

The above expression is valid for the anti-particle by replacing u with $\bar{v}$.
For theories like QED that have particles at $m = 0$ (photon), the brunch cut starts immediately with the pole. Theories with unstable particles may also need more conditions.

7 Regularization

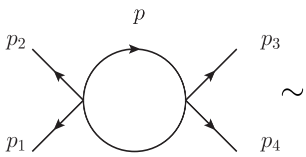
To study the renormalizability of the $\lambda\phi^4$ theory we must be able to manipulate the divergences that occur. So we need some procedure that makes the divergent momentum integration finite but leaves the convergent contribution unaffected.

The simplest method is to cutoff the Euclidean momentum. We have seen integrals like



$$\sim \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{7.1}$$

and



$$\sim \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(p_1 + p_2 + p)^2 - m^2 + i\epsilon}\,. \tag{7.2}$$

To put the second integral in the form of the first one we can use *Feynman parametrization*:

$$\frac{1}{AB} = \int_0^1 dx\, \frac{1}{[Ax + B(1-x)]^2} = \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{[Ax+By]^2}\,. \tag{7.3}$$

A more general formula for more denominators reads

$$\frac{1}{A_1\, A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n\, \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{(x_1A_1 + \cdots + x_nA_n)^n}\,. \tag{7.4}$$

By differentiating (7.4) with respect to A_1 we find:

$$\frac{1}{A_1^2\, A_2 \cdots A_n} = n! \int_0^1 dx_1 \cdots dx_n\, \delta\left(\sum_i x_i - 1\right) \frac{x_1}{(x_1A_1 + \cdots + x_nA_n)^{n+1}}\,, \tag{7.5}$$

and this formula has the advantage that one less Feynman parameters is needed for the case where there are two identical factors in the denominator, and in particular we have

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^\alpha (1-x)^{\beta-1}}{[Ax + B(1-x)]^{\alpha+\beta}}\,. \tag{7.6}$$

(See Appendix E for definition of the gamma function $\Gamma(z)$.) Using (7.3), and defining $P = p_1 + p_2$, we obtain for (7.2):

$$\begin{aligned} &\frac{i}{p^2 - m^2 + i\epsilon} \frac{1}{(P+p)^2 - m^2 + i\epsilon} = \int_0^1 dx\, \frac{1}{[xp^2 - xm^2 + (1-x)(p+P)^2 + (x-1)m^2]^2} \\ &= \int_0^1 dx\, \frac{1}{[p^2 - m^2 + 2(1-x)P\cdot p + (1-x)P^2]^2}\,, \end{aligned} \tag{7.7}$$

where in the denominator we now build a square by adding and subtracting some terms:

$$\begin{aligned} &xp^2 - xm^2 + P^2 + p^2 + 2p\cdot P - xP^2 - xp^2 - 2P\cdot px + xm^2 - m^2 \\ &= p^2 - m^2 + 2(1-x)P\cdot p + (1-x)P^2 \end{aligned}$$

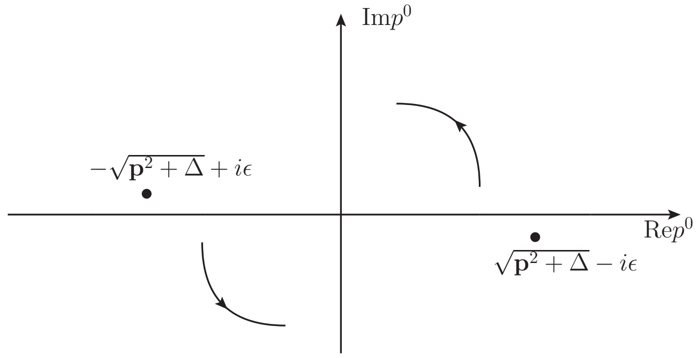


Figure 7.1: Pole of the propagators and Wick rotation.

$$\begin{aligned}
&= (p+(1-x)P)^2 - m^2 + 2xP^2 - x^2P^2 - xP^2 \\
&= (p+(1-x)P)^2 - m^2 + P^2x(1-x) \\
&= p' - \Delta\,, \qquad (7.8)
\end{aligned}$$

where we have used the following definition (one is a shift in the momentum integration):

$$\begin{cases} \Delta = -P^2x(1-x)+m^2\,, \\ p' = (p+(1-x)P)\,. \end{cases} \qquad (7.9)$$

So we have to calculate in general integrals of the form

$$I = \int_0^1 dx \int \frac{d^4p}{(2\pi)^4}\frac{1}{[p^2-\Delta]^2}\,, \text{in our case}\, n=2\,, \qquad (7.10)$$

and it has poles in $\pm\sqrt{\mathbf{p}^2+\Delta}\mp i\epsilon$, as shown in figure 7.1. We perform therefore a Wick rotation: $p^0 = ip_E^0 \Rightarrow p_E^0 = -ip^0$, that brings to

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{[p^2-\Delta]^n} = \frac{i}{(-1)^n}\frac{1}{(2\pi)^4}\int d^4p_E \frac{1}{[p_E^2+\Delta]^n} = \frac{i(-1)^n}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\rho \frac{\rho^3}{[\rho^2+\Delta]^n} \qquad (7.11)$$

and for $n=2$ the integral diverges. (For definition of the solid angle $d\Omega_4$ see Appendix E.) We can think to regularize the integral by introducing a cutoff:

$$\int_0^\Lambda d\rho \frac{\rho^3}{[\rho^2+\Delta]^n}\,. \qquad (7.12)$$

In this cutoff procedure for very large Λ the convergent integral will receive a negligible contribution from the region with ρ larger than Λ, so they are unaffected, whereas diagrams quadratically or logarithmically divergent will have terms proportional to Λ^2 or $\ln\Lambda$. We can use also a "covariant cutoff" which was common up to 1972, which is called *Pauli-Villars regularization.*

The Pauli-Villars regularization consists in substituting the free propagator as follows (introducing a fictitious field of mass Λ)

$$\frac{1}{p^2-m^2} \to \frac{1}{p^2-m^2} - \frac{1}{p^2-\Lambda^2} = \frac{m^2-\Lambda^2}{(p^2-m^2)(p^2-\Lambda^2)}\,, \qquad (7.13)$$

with large Λ. Then for values $p \ll \Lambda$ the integration is unaffected. Convergent diagrams receive a negligible contribution from the region where p is large so they are untouched. On the other hand diagrams that were divergent are now convergent to the additional factor $1/p^2$ supplied by the regularization for $|p| \geq \Lambda$.

Both cutoff and Pauli-Villars regularization have a big drawback if they are used in a gauge theory. The problem is that they break gauge invariance. So one should look at a different method and this is *dimensional regularization*, 't Hooft and Veltman (1972), Bollini and Gianbiagi (1972), Cicuta and Mantoldi (1972).

7.1 Dimensional Regularization

The idea of dimensional regularization is based on the following: up to now we tried to make the integrals finite by increasing the powers of momentum p in the denominator of the integrand. Another way to regularize the integral would be to decrease the power of p deriving from the volume element at numerator. Typically we have d^4p because we are in four-dimensional space-time continuum. To decrease the contribution from the volume element would require us to work in a space-time continuum of dimension $D < 4$ (typically we will use $D = 4 - \varepsilon$). This is called dimensional regularization method. Dimensional reguralization preserves all symmetries apart form dilatation invariance because in $D = 4$ the coupling constant is dimensionless and the chiral invariance due to the difficulties of defining γ^5 in D-dimensions.

Indeed we have integrals of the form

$$I_4(k) = \int_{-\infty}^{+\infty} d^4p F(p, k)\,, \tag{7.14}$$

where for large p one has $F \sim 1/p^2$ or $1/p^4$. For instance let us consider the case that $F \sim 1/p^4$ at large momenta, then in $D = 4$ the integral diverges, however in $D = 2$ the integral converges in the UV.

Mathematically we can introduce the function

$$I(D, k) = \int d^D p F(p, k)\,, \tag{7.15}$$

as a function of the (complex) variable D. We can evaluate it in a domain where I has no singularity in the D-plane. Then consider a function $I'(D, k)$ that coincides with I in the domain of convergence of (7.15) in the D-plane and has well-defined singularities outside the domain of convergence. We say that by analytic continuation I and I' are the same function.

So with this method we consider the whole theory in D space-time dimensions. This change has to occur already at the level of the Lagrangian, and consequently one finds that the coupling constants which are dimensionless in four dimensions become dimensionful in an arbitrary D. Moreover the integrals and the Green's functions will depend on D, and those corresponding to divergent contributions in four dimensions typically have poles in $D = 4$.

For $\lambda\phi^4$ this means that we have to parametrize the $\mathcal{L}$ to D dimensions:

$$\begin{cases} S = \int d^D x \mathcal{L} \;\Rightarrow\; [\mathcal{L}] = m^D\,, \\ [\partial^\mu \phi_B \partial_\mu \phi_B] = m^D \;\Rightarrow\; [\phi_B] = m^{\frac{D-2}{2}} \;\Rightarrow\; [\phi_B^4] = m^{2D-4}\,, \\ [\lambda] = m^{D-2D+4} = m^{4-D}\,. \end{cases}$$

So the coupling constant λ should be multiplied by a factor μ^{4-D} where μ is an arbitrary mass parameter:

$$\mathcal{L}_B = \frac{1}{2}(\partial^\mu \phi_B)(\partial_\mu \phi_B) - \frac{1}{2} m_B^2 \phi_B^2 - \mu^{4-D} \frac{\lambda_B}{4!} \phi_B^4\,. \tag{7.16}$$

Now we can use DR to calculate the first divergent contribution to $\Gamma^{(2)}$ (or $G^{(2)}$) which is a primitive divergence:

$$\frac{\lambda_B}{2} D_F(0) = \frac{\lambda_B}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_B^2 + i\eta} = \frac{\lambda_B}{2} \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m_B^2}\,.$$

In dimensional regularization we have

$$\frac{i}{2} \lambda_B \mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - m_B^2 + i\eta}\,. \tag{7.17}$$

The integral is in D-dimension Minkowski space, we change the $i\epsilon$ prescription notation in $i\eta$ to avoid the ambiguities with $D = 4 - \varepsilon$.

In the exercise sheet 6 you demonstrate (making a Wick rotation first) that

$$\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - m_B^2 + i\eta} = -i \frac{\mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) (m_B^2)^{D/2-1}\,, \tag{7.18}$$

and in general we have

$$\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 - m_B^2 + i\eta)^n} = (-1)^n i \frac{\mu^{4-D}}{(4\pi)^{D/2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} (m_B^2)^{D/2-n}\,, \tag{7.19}$$

or again written more in general

$$\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 - \Delta + i\eta)^n} = (-1)^n i \frac{\mu^{4-D}}{(4\pi)^{D/2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} (\Delta)^{D/2-n}\,. \tag{7.20}$$

We notice that odd powers in p^μ at the numerator gives zero. Moreover $p^\mu p^\nu$ in the numerator gives $g^{\mu\nu} p^2/D$ and

$$\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{p^2}{(p^2 - m_B^2 + i\eta)^n} = (-1)^{n-1} i \frac{\mu^{4-D}}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma\left(n - \frac{D}{2} - 1\right)}{\Gamma(n)} (m_B^2)^{D/2-n+1}\,, \tag{7.21}$$

$$\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^2}{(p^2 - m_B^2 + i\eta)^n} = (-1)^{n-1} i \frac{\mu^{4-D}}{(4\pi)^{D/2}} \frac{D(D+2)}{4} \frac{\Gamma\left(n - \frac{D}{2} - 2\right)}{\Gamma(n)} (m_B^2)^{\frac{D}{2}-n+2}\,, \tag{7.22}$$

In particular we point out that *scaleless integrals are zero in DM* (see exercise sheet 6)

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^\alpha} = 0\,. \tag{7.23}$$

For our integral in (7.17) we obtain

$$i\mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - m_B^2 + i\eta} = i(-i) \frac{\mu^{4-D} m_B^{D-2}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right)\,. \tag{7.24}$$

The expression in (7.24) may be used to define the integral by analytic continuation in D, in regions where the original integral does not exist. The factor $\Gamma(1 - D/2)$ has poles in $D = 2, 4, 6, ...$ and of course we are interested in the vicinity of the pole $D = 4$. To investigate this we put $D = 4 - \varepsilon$ and we get

$$\begin{cases} 4 - D = \varepsilon\,, \\ D - 2 = 2 - \varepsilon\,, \\ \frac{D}{2} = 2 - \frac{\epsilon}{2}\,, \\ 1 - \frac{D}{2} = 1 - 2 + \frac{\varepsilon}{2} = -1 + \frac{\varepsilon}{2}\,. \end{cases} \tag{7.25}$$

Expanding around $D = 4$ and using $a^\varepsilon = e^{\varepsilon \ln a} \simeq 1 + \varepsilon \ln a + \mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{cases} \mu^\varepsilon \simeq 1 + \varepsilon \ln \mu + \mathcal{O}(\varepsilon^2)\,, \\ m_B^{2-\varepsilon} = m_B^2 m_B^{-\varepsilon} = m_B^2 (1 - \varepsilon \ln m_B + \mathcal{O}(\varepsilon^2))\,, \\ (4\pi)^{\frac{D}{2}} = (4\pi)^2 (4\pi)^{-\frac{\varepsilon}{2}} = (4\pi)^2 \left(1 - \frac{\varepsilon}{2} \ln(4\pi) + \mathcal{O}(\varepsilon^2)\right)\,, \\ \Gamma\left(1 - \frac{D}{2}\right) = \Gamma(-1 + \frac{\varepsilon}{2}) = (-1)\left[\frac{2}{\varepsilon} + 1 - \gamma_E + \mathcal{O}(\varepsilon)\right]\,. \end{cases}$$

The last expression comes from the more general one

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma_E\right) + \mathcal{O}(\varepsilon)\right]\,. \tag{7.26}$$

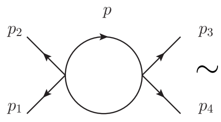
Putting everything together we obtain from (7.24) (leaving out one factor i)

$$\begin{aligned} &(-i)(1 + \varepsilon \ln \mu + \mathcal{O}(\varepsilon^2)) m_B^2 (1 - \varepsilon \ln m_B + \mathcal{O}(\varepsilon^2)) \frac{1}{(4\pi)^2} \left(1 + \frac{\varepsilon}{2} \ln(4\pi) + \mathcal{O}(\varepsilon^2)\right) \\ &\times \left[-\frac{2}{\varepsilon} - 1 + \gamma_E + \mathcal{O}(\varepsilon)\right] \\ &= (+i) \frac{m_B^2}{(4\pi)^2} \left[\frac{2}{\varepsilon} + 1 - \gamma_E + 2\ln\mu - 2\ln m_B + \ln(4\pi)\right] \\ &= i \frac{m_B^2}{(4\pi)^2} \left[\frac{2}{\varepsilon} + 1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_B^2}\right)\right]\,, \end{aligned} \tag{7.27}$$

so that

$$\begin{aligned} &\frac{i}{2} \lambda_B \mu^{4-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - m_B^2 + i\eta} \\ &= i \left\{ i \frac{\lambda_B m_B^2}{16\pi^2} \frac{1}{\varepsilon} + i \frac{\lambda_B m_B^2}{32\pi^2} \left[1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_B^2}\right)\right] \right\}\,, \end{aligned} \tag{7.28}$$

where the first term embeds the divergence and the second one contains the finite terms which depends on the arbitrary scale μ.
Now let us calculate the four-point function at order λ_B^2:



$$\sim \frac{\lambda_B^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m_B^2+i\eta}\frac{1}{(p_1+p_2+p)^2-m_B^2+i\eta}\,, \tag{7.29}$$

and using the Feynman parametrization we have already calculated and with the following substitution

$$\begin{cases}\Delta = -P^2x(1-x)+m_B^2\,,\\ P = p_1+p_2\,,\end{cases} \tag{7.30}$$

we have for eq. (7.29)

$$\frac{\lambda_B^2}{2}(\mu^2)^{4-D}\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\frac{1}{[p^2-\Delta]^2}\,. \tag{7.31}$$

Then using eq. (7.19) we get

$$\begin{aligned}&\frac{\lambda_B^2}{2}(\mu^2)^{4-D}\int_0^1 dx\frac{(-1)^2}{(4\pi)^{D/2}}(i)\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(2)}\Delta^{\frac{D}{2}-2}\\ &= i\frac{\lambda_B^2}{2\cdot 16\pi^2}(\mu^2)^{2-\frac{D}{2}}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 dx\left[\frac{-P^2x(1-x)+m_B^2}{4\pi\mu^2}\right]^{\frac{D}{2}-2}\,.\end{aligned} \tag{7.32}$$

In the limit $D\to 4$, using the useful relations

$$\begin{cases}\Gamma\left(2-\frac{D}{2}\right)=\Gamma\left(\frac{\varepsilon}{2}\right)=\frac{2}{\varepsilon}-\gamma_E+\mathcal{O}(\varepsilon)\,,\\ 2-\frac{D}{2}=\frac{\varepsilon}{2}\,,\\ \left(\frac{\Delta}{4\pi\mu^2}\right)^{-\frac{\varepsilon}{2}}\simeq 1-\frac{\varepsilon}{2}\ln\left(\frac{\Delta}{4\pi\mu^2}\right)\,,\end{cases} \tag{7.33}$$

we obtain for eq. (7.32)

$$\begin{aligned}&i\frac{\lambda_B^2}{32\pi^2}\mu^\varepsilon\left(\frac{2}{\varepsilon}-\gamma_E+\mathcal{O}(\varepsilon)\right)\left\{1-\frac{\varepsilon}{2}\int_0^1 dx\ln\frac{m_B^2-P^2x(1-x)}{4\pi\mu^2}\right\}\\ &= i\frac{\lambda_B^2}{16\pi^2}\frac{\mu^\varepsilon}{\varepsilon}-i\frac{\lambda_B^2\mu^\varepsilon}{32\pi^2}\left\{\gamma_E+\int_0^1 dx\ln\frac{m_B^2-P^2x(1-x)}{4\pi\mu^2}\right\}\,,\end{aligned} \tag{7.34}$$

where we note the first term that embeds the divergence in $1/\varepsilon$ and the second term that is finite and depends on $s=P^2=(p_1+p_2)^2$.

Notice:
The loop integral we have considered has the property that after the UV divergence is regulated by DR the remaining Feynman parameters integration is convergent when $D\to 4$. This is always the case for any diagram encountered in $\lambda\phi^4$. However this will be not always true for gauge theories. In fact the gauge field (photons, gluons) have zero mass. The presence of zero-mass particles generates additional divergences in the Feynman integrals which comes from the behaviour of the integrand at small p (infrared divergences =

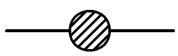
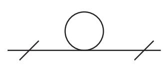
IR). This can be illustrated in $\lambda\phi^4$ setting $m_B = 0$. Then the integral in (7.29) is also IR divergent when $p_1 + p_2 = 0$. Both UV and IR divergences are manifest in (7.29). In our DR formalism, where the integral over the momenta p is taken, the UV divergence shows itself as the pole of $\Gamma(2 - D/2)$ in (7.32) at $D = 4$, while the IR divergence is contained in the remaining factors of the integrand, which is $\propto (P^2)^{(D/2-2)}$, and it diverges as $P^2 \to 0$ when $\Re(D/2 - 2) < 0$.

The IR divergences are less "serious" than the UV ones since at least in QED they do not contribute to observable quantities. In any case they have to be regulated to have well defined quantities to deal with. One option is to introduce a mass $m_{\rm IR}$ for the massless particle and at the end let $m_{\rm IR} \to 0$. However this may give problems with gauge invariance. A better option is to maintain $D \neq 4$ in the Feynman parameter integration. After that the UV divergences have been removed by renormalization we are free to continue from $D < 4$ to $D > 4$ which regulates the remaining infrared divergence. This means that we are using dimensional regularization also for IR divergences.

Now we can write the 1PI functions $\Gamma^{(2)}(p)$ and $\Gamma^{(4)}(p)$. We start with

$$\Gamma_B^{(2)}(p) = p^2 - m_B^2 - \Sigma(p)\,, \tag{7.35}$$

that diagrammatically was

 $= -i\Sigma(p) =$  $+ \cdots$ (7.36)

so that

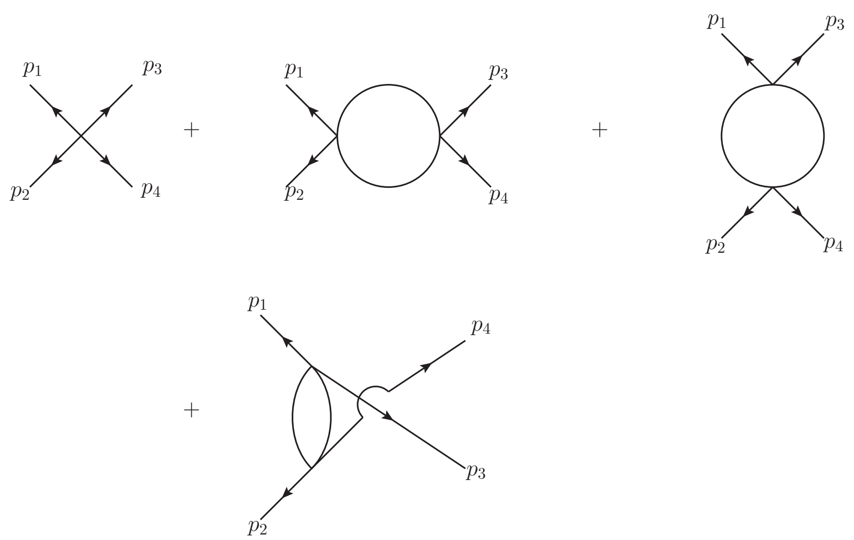
$$-\,i\Sigma(p) = i\frac{\lambda_B m_B^2}{16\pi^2}\frac{1}{\varepsilon} + i\frac{\lambda_B m_B^2}{32\pi^2}\left[1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_B^2}\right)\right]\,, \tag{7.37}$$

and therefore

$$\Sigma(p) = -\frac{\lambda_B m_B^2}{16\pi^2}\frac{1}{\varepsilon} + \text{finite terms,} \tag{7.38}$$

$$\Gamma_B^{(2)}(p) = p^2 - m_B^2\left(1 - \frac{\lambda_B}{16\pi^2\varepsilon}\right) + \text{finite terms.} \tag{7.39}$$

For the four-point Green's function, $i\Gamma^{(4)}(p_1, p_2, p_3, p_4)$ we have the following diagrammatic representation, at order λ^2



(7.40)

and we can think of the three diagrams at order λ^2 in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2 \,, \tag{7.41}$$
$$t = (p_1 + p_3)^2 \,, \tag{7.42}$$
$$u = (p_1 + p_4)^2 \,, \tag{7.43}$$

for the s, t and u channel. We can define

$$F(s, m_B, \mu) = \int_0^1 dx \ln \frac{m_B^2 - sx(1-x)}{4\pi\mu^2} \,, \tag{7.44}$$

then from eq. (7.34) we have

$$\begin{aligned} i\Gamma_B^{(4)}(p_1, p_2, p_3, p_4) &= -i\lambda_B\mu^\varepsilon + \frac{3i\lambda_B^2\mu^\varepsilon}{16\pi^2\varepsilon} \\ -i\frac{\lambda_B^2\mu^\varepsilon}{32\pi^2}&(3\gamma_E + F(s, m_B, \mu) + F(t, m_B, \mu) + F(u, m_B, \mu)) \\ = -i\lambda_B\mu^\varepsilon &\left(1 - 3\frac{\lambda_B}{16\pi^2\varepsilon}\right) + \text{finite terms} \,, \end{aligned} \tag{7.45}$$

and $\Gamma_B^{(2)}$ *and* $\Gamma_B^{(4)}$ *has to be made finite.*

8 γ matrices and dimensional regularization

Dimensional regularization (DR) does not preserve dilatation invariance (the coupling constants are no longer dimensionless) and chiral invariance due to the difficulty to define the γ_5 in D-dim.
When one extends the theory to a D-dimensional space, the fields in the Lagrangian and the coupling constants acquire different dimensions. For fermions we have also that the Dirac matrices have to be extended to D-dimensions. In D-dimensions we will have a set of D γ-matrices:

$$\gamma_\mu\,, \quad \mu = 0, 1, ..., D-1\,, \tag{8.1}$$

and we require that they satisfy the Clifford Algebra:

$$\begin{cases} \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1} \\ g_{\mu\nu} = ||g_{\mu\nu}|| = diag(+1, -1, ..., -1) \end{cases} \tag{8.2}$$

with the following properties:

$$\gamma^{0\dagger} = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i \qquad i = 1, ..., D-1 \tag{8.3}$$

When we use DR the Lorentz indices range over an infinite set of values, so we need infinite-dimensional matrices to represent the algebra (8.2). We need also to define a trace operation:

$$\mathrm{tr}\mathbb{1} = f(D) \tag{8.4}$$

so that the representation behaves as if its dimensions were $f(D)$.
For $D = 4$ we have $f(4) = 4$.
In even integer dimension the standard representation of the γ_μ's has dimension $2^{D/2}$. However it is not necessary to choose $f(D) = 2^{D/2}$. We can always choose $f(D) = 4 \Rightarrow$ other choices lead to constant terms that go away with renormalization (any change in $f(D)$ amounts to a renormalization-group transformation).
Since at the end of the calculation we can always choose $f(D) = 4$ and $f(D)$ factorizes this is not relevant. However in the evaluation of diagrams there might be additional contributions of order ε ($D = 4 - \varepsilon$). In particular extra terms in ε in the contraction identities can contribute to the final result if they multiply a factor $1/\varepsilon$ from a divergent integral.

Contraction identities

Standard manipulations involving the anticommutation relations are valid independently of D. We have e.g.:

$$\begin{cases} \gamma^\mu\gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\}\,\mathbb{1} = g^\mu_\mu\mathbb{1} = D\mathbb{1} \\ \gamma^\mu\gamma_\nu\gamma_\mu = \gamma^\mu\{\gamma_\nu, \gamma_\mu\} - \gamma^\mu\gamma_\mu\gamma_\nu = 2g_{\mu\nu}\gamma^\mu - \gamma^\mu\gamma_\mu\gamma_\nu = (2-D)\gamma_\nu \end{cases} \tag{8.5}$$

and so on:

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\mu\lambda}\mathbb{1} + (D-4)\gamma^\nu\gamma^\lambda\,. \tag{8.6}$$

One needs also to evaluate traces of products of γ's.
The trace of a matrix is linear and cyclic

$$\begin{cases} \mathrm{tr}(aA + bB) = a\mathrm{tr}(A) + b\mathrm{tr}(B)\,, \\ \mathrm{tr}(AB) = \mathrm{tr}(BA)\,, \end{cases} \tag{8.7}$$

where A and B are any product of γ-matrices and a and b are numbers. These properties together with the value of $\mathrm{tr}\mathbb{1}$ define the trace of any linear combination of products of γ-matrices. It turns out that traces that do not involve γ^5 are independent of dimensionality:

$$\begin{aligned} \mathrm{tr}(\gamma^\mu\gamma^\nu) &= \mathrm{tr}(\gamma^\nu\gamma^\mu) && \text{cyclicity} \\ &= \mathrm{tr}(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\mathbb{1}) && \text{anticommutation} \\ &= -\mathrm{tr}(\gamma^\mu\gamma^\nu) + 2g^{\mu\nu}\mathrm{tr}\mathbb{1} && \text{linearity} \\ \Rightarrow \mathrm{tr}(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\mathrm{tr}\mathbb{1} && (\mathrm{tr}\mathbb{1} = 4)\,. \end{aligned} \tag{8.8}$$

In $D = 3$ the γ matrices are represented by the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.9}$$

In the same way one obtains:

$$\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) = (g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu} + g^{\kappa\nu}g^{\lambda\mu})\mathrm{tr}\mathbb{1}\,. \tag{8.10}$$

The trace of the products of an odd number of γ-matrices is zero.
For example:

$$\begin{aligned} D\,\mathrm{tr}\gamma^\lambda &= \mathrm{tr}(\gamma^\kappa\gamma_\kappa\gamma^\lambda) = -\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma_\kappa) + 2\mathrm{tr}\gamma^\lambda \\ &= -\mathrm{tr}(\gamma_\kappa\gamma^\kappa\gamma^\lambda) + 2\mathrm{tr}\gamma^\lambda = -D\,\mathrm{tr}\gamma^\lambda + 2\mathrm{tr}\gamma^\lambda \\ \Rightarrow\ (2D-2)\mathrm{tr}\gamma^\lambda &= 0 \\ \Rightarrow\ \mathrm{tr}\gamma^\lambda &= 0\,. \end{aligned} \tag{8.11}$$

Definition of γ_5

First of all we notice that $\gamma_5 = \gamma^5$. In $D = 4$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,, \tag{8.12}$$

with the properties:

$$\begin{cases} (\gamma^5)^\dagger = \gamma^5,\ (\gamma^5)^2 = 1,\ \{\gamma^5, \gamma^\mu\} = 0\,, \\ \mathrm{tr}\gamma^5 = 0,\ \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu) = 0\,, \end{cases} \tag{8.13}$$

with $\varepsilon_{0123} = 1$, $\varepsilon_{\mu\nu\rho\sigma}$ is totally antisymmetric Lorentz invariant tensor (Levi-Civita).
It is not simple to generalize this definition with these properties to the case of D dimensions. We need γ_5 to define for example the axial current $\bar\psi\gamma^\mu\gamma_5\psi$.
In $D = 4$ we have the trace formula:

$$\mathrm{tr}\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = i\varepsilon^{\kappa\lambda\mu\nu}\,\mathrm{tr}\mathbb{1} = -i\varepsilon_{\kappa\lambda\mu\nu}\,\mathrm{tr}\mathbb{1}\,. \tag{8.14}$$

If we generalize the definition of γ^5 to a D-dim space writing

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3...\gamma^{D-1}\,, \tag{8.15}$$

then γ^5 commutes with all γ matrices if D is odd and anticommutes with them if D is even. But, this does not allow an interpretation for non-integer values of D.
If we define γ^5 as a matrix such that $\{\gamma^\mu, \gamma^5\} = 0$ holds, then there is no such matrix that can be written in terms of the γ's and anticommutes for all natural values of D.
Since however this does not depend on an explicit realization, a practical method can still be developed along these lines: in an expression containing several γ_5 on one quark line, one moves all of them to the end of the line (using the anticommutation rule). If the number of γ_5 is even, they will cancel ($(\gamma^5)^2 = 1$); if the number is odd, one γ^5 will remain at the end and the method may lead to ambiguites for closed loops (these can be resolved, at least up to two loops with the help of the γ_5 anomaly).
Or we can have the 't Hooft-Veltman solution:

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \varepsilon_{0123} = 1\,. \tag{8.16}$$

Then we have:

$$\begin{cases} \{\gamma_5, \gamma^\mu\} = 0, \text{ if} & \mu = 0,1,2,3 \\ [\gamma_5, \gamma^\mu] = 0, & \text{otherwise} \\ (\gamma^5)^2 = 1\,, & (\gamma_5)^\dagger = \gamma^5\,. \end{cases} \tag{8.17}$$

This presents no problem at any order in perturbation theory, however it lacks full Lorentz invariance.

9 Renormalization

Renormalization is the procedure necessary in any interacting field theory that enables us to remove the infinities by absorbing them into the renormalization constants. We start from the bare Lagrangian

$$\mathcal{L}_B = \frac{1}{2}(\partial^\mu \phi_B)(\partial_\mu \phi_B) - \frac{1}{2} m_B^2 \phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4 \,, \tag{9.1}$$

and we define a renormalized field $\phi(x)$ by:

$$\phi_B(x) = Z_\phi^{\frac{1}{2}} \phi(x) \,, \tag{9.2}$$

where $\phi(x) = \phi_{\text{ren}}(x)$ and the wave function renormalization constant Z_ϕ differs from unity due to quantum corrections:

$$Z_\phi = 1 + \delta Z_\phi \,. \tag{9.3}$$

In the same way quantum effects lead to mass and coupling constant renormalization:

$$Z_\phi m_B^2 = m^2 + \delta m^2 \,, \tag{9.4}$$

$$Z_\phi^2 \lambda_B = \lambda + \delta\lambda \,, \tag{9.5}$$

where m is the renormalized mass and λ is the renormalized coupling constant in the above equations. Note that when we use dimensional regularization the coupling constant λ will be dimensionful, however one can keep λ dimensionless and instead multiply it with some powers of μ. Here we keep both λ and λ_B dimensionless and we write Eq (9.5) as

$$Z_\phi^2 \lambda_B \mu^\varepsilon = \lambda \mu^\varepsilon + \delta\lambda \,. \tag{9.6}$$

(Note that with the above choice $\delta\lambda$ is a dimensionful quantity proportional to μ^ε.) Then we can write

$$\mathcal{L}_B = \mathcal{L} + \delta\mathcal{L} \,, \tag{9.7}$$

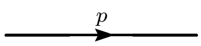
$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) - \frac{\lambda}{4!}\phi^4 \,, \tag{9.8}$$

and $\delta\mathcal{L}$ is the "counter term" Lagrangian

$$\delta\mathcal{L} = \frac{1}{2}\delta Z_\phi \left(\partial_\mu \phi \partial^\mu \phi\right) - \frac{1}{2}\delta m^2 \phi^2 - \frac{\delta\lambda}{4!}\phi^4 \,. \tag{9.9}$$

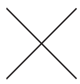
The Feynman rules for $\mathcal{L}$ are the ones we have already obtained:

1) for the propagator



$$= \frac{i}{p^2 - m^2 + i\epsilon} \tag{9.10}$$

2) for the vertex



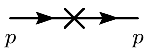
$$= -i\lambda \tag{9.11}$$

The $\delta\mathcal{L}$ generates extra vertices. In fact we can write

$$\begin{cases} \mathcal{L} + \delta\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \,, \\ \mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 + \delta\mathcal{L} \,. \end{cases} \tag{9.12}$$

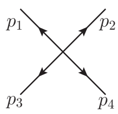
Then using the same procedure we have developed before to obtain Feynman rules (obtain $G^{(n)}$ as derivatives of Z_{free}) we can obtain the additional rules:

1) there is a vertex involving two lines. Momentum is conserved and the vertex is associated with a factor (the i comes from the action):



$$= i(\delta Z_\phi \, p^2 - \delta m^2), \tag{9.13}$$

2) there is an additional vertex involving four lines associated to a factor $-i\delta\lambda$



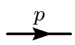
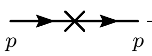
$$= -i\delta\lambda \,, \tag{9.14}$$

where $p_1 + p_2 + p_3 + p_4 = 0$. The last Feynman rule follow directly from the rule valid for the vertex in (9.11) for the renormalized Lagrangian. We can check that the rule for the first counter term in (9.13) is correct in a simple way. Let us set $\lambda = \delta\lambda = 0$ temporarily. Then we have a free theory and clearly

$$G_0^{(2)}(p) = \frac{i}{p^2(1 + \delta Z_\phi) - (m^2 + \delta m^2) + i\epsilon} \,, \tag{9.15}$$

and we expand the right-hand side to the lowest order in δZ and δm^2 we have:

$$G_0^{(2)}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} i(\delta Z_\phi p^2 - \delta m^2) \frac{i}{p^2 - m^2 + i\epsilon} + \cdots$$

=  +  + ⋯ , (9.16)

as required by the rule in (9.13). The quantities δm^2 and δZ_ϕ are non-zero because of quantum effects due to interactions. (δZ_ϕ gets no contributions at one loop: this is a special feature of $\lambda\phi^4$ theory which does not occur in more general theories of scalar fields, for example in Yukawa theory it is not zero). Thus we may expand them as power series in λ starting at order λ

$$\delta m^2 = \sum_{i=1}^{\infty} \delta m_i^2 \,, \tag{9.17}$$

$$\delta Z_\phi = \sum_{i=1}^{\infty} \delta Z_i \,, \tag{9.18}$$

with δm_i and δZ_i proportional to λ^i. In the same way since $\delta\lambda/\lambda$ is a quantum effect of order λ caused by the interaction, we may write $\delta\lambda$ as a power series starting at λ^2

$$\delta\lambda = \sum_{i=2}^{\infty} \delta\lambda_i \,. \tag{9.19}$$

We can now calculate the proper Green's functions $\tilde{\Gamma}^{(n)}$ as functions of the external momenta, the renormalized parameters m^2, λ and the counter terms parameters δZ_ϕ, δm^2, $\delta\lambda$.

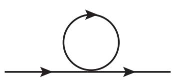
For example $\tilde{\Gamma}^{(2)}(p)$ is given at leading order by:

$$\tilde{\Gamma}^{(2)}(p) = p^2 - m^2 - \Sigma(p)\,, \quad \tilde{\Gamma}^{(2)}(p)\tilde{G}^{(2)}(p) = i\,. \tag{9.20}$$

Previously we calculated

$$\Sigma(p) = -\left(\frac{1}{i}\right)(-i)\frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = \frac{\lambda}{2}D_F(0)\,, \tag{9.21}$$

but now we have a new contribution at order λ induced by the counterterms. In a diagrammatic form we have

$-i\Sigma(p) = \ +\frac{1}{2}$  $+$  $+\mathcal{O}(\lambda^2)$. (9.22)

The first diagram has been already calculated and it is given in Eq. (9.21), while the last one comes from the counterterms. Putting everything together we obtain

$$\tilde{\Gamma}^{(2)}(p) = p^2(1+\delta Z_1) - \left[\left(m^2 + \frac{1}{2}\lambda D_F(0)\right) + \delta m_1^2\right] + \mathcal{O}(\lambda^2)\,. \tag{9.23}$$

From our previous calculations we know that

$$\begin{aligned} \frac{1}{2}\lambda D_F(0) &= i\frac{\lambda}{2}\mu^{4-D}\int \frac{d^Dp}{(2\pi)^D}\frac{i}{p^2-m^2+i\epsilon} \\ &= -\frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} - \frac{\lambda m^2}{32\pi^2}\left[1-\gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]\,. \end{aligned} \tag{9.24}$$

So we have

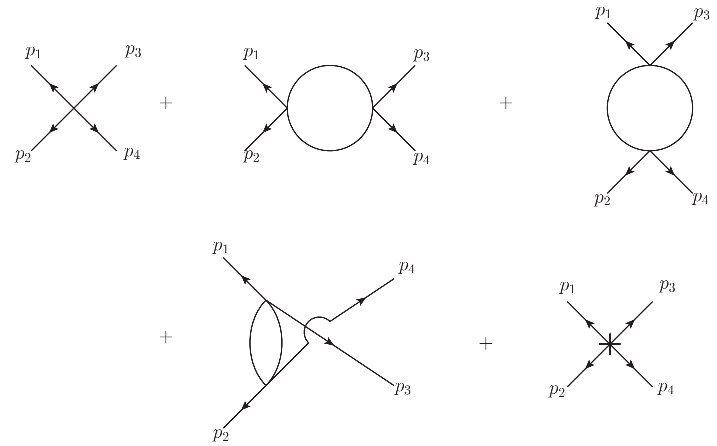
$$\tilde{\Gamma}^{(2)}(p) = p^2(1+\delta Z_1) - m^2 - \delta m_1^2 + \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} + \frac{\lambda m^2}{32\pi^2}\left[1-\gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]\,. \tag{9.25}$$

The quantity δm_1^2 and δZ_1 are fixed by a renormalization scheme which is essentially a bounday condition (renormalization condition) on $\tilde{\Gamma}^{(2)}$. We will discuss in a moment the most used renormalization schemes. The point shared by *all schemes* is that since $\tilde{\Gamma}^{(2)}$ is an observable quantity it has to be finite in $D=4$. So for $D \to 4$ we should have

$$\begin{cases} \delta m_1^2 - \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} \to \text{constant} \\ \delta Z_1 \to \text{constant}\,. \end{cases} \tag{9.26}$$

Since the constants have not been fixed (this will be done by the particular renormalization scheme), the finite parts of δm_1^2 and δZ_1 are not constrained.

In the same way we can evaluate $\tilde{\Gamma}^{(4)}$ to order λ^2: we list the diagrammatic expression of $i\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$ as follows



(9.27)

We have already calculated the first four diagrams and now we have

$$\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = -\lambda\mu^{\varepsilon} + \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}$$
$$-\frac{\lambda^2\mu^{\varepsilon}}{32\pi^2}(3\gamma_E + F(s, m, \mu) + F(t, m, \mu) + F(u, m, \mu)) - \delta\lambda_2 + \mathcal{O}(\lambda^3)\,, \tag{9.28}$$

with

$$F(s, m, \mu) = \int_0^1 dx \ln\left(\frac{m^2 - x(1-x)s}{4\pi^2\mu^2}\right) \tag{9.29}$$

Again since $\tilde{\Gamma}^{(4)}$ has to be finite in any renormalization scheme then

$$\frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon} - \delta\lambda_2 \to \text{constant}\,. \tag{9.30}$$

Also in this case the finite part of $\delta\lambda_2$ is arbitrary. It will be fixed by the renormalization scheme we adopt. Therefore different schemes correspond to different choices of the finite parts of the counterterms and thus to different choices of the renormalized parameters.

9.1 Relation between the bare and renormalized proper functions

In $\lambda\phi^4$ the only proper functions that are divergent at one loop are $\tilde{\Gamma}^{(2)}_B$ and $\tilde{\Gamma}^{(4)}_B$: we want to relate them to the renormalized $\tilde{\Gamma}^{(2)}$ and $\tilde{\Gamma}^{(4)}$ which we have shown are finite at one loop. Then, using $\phi_B(x) = \sqrt{Z_\phi}\phi(x)$ and $\mathcal{L}_B = \mathcal{L} + \delta\mathcal{L}$, we obtain

$$\mathcal{L} + \delta\mathcal{L} + J(x)\phi(x) = \mathcal{L}_B + J_B(x)\phi_B(x)\,, \tag{9.31}$$

with

$$J_B(x) = Z_\phi^{-\frac{1}{2}} J(x)\,. \tag{9.32}$$

We may use this to relate the generating functional of the bare and renormalized theories. Let us denote with $Z_B[J_B]$ the generating functional with Lagrangian $\mathcal{L}_B$ and source J_B and functional integration variable ϕ_B; it reads

$$Z_B[J_B] = \frac{1}{N}\int \mathcal{D}\phi_B\, exp\left\{i\int d^4x\,(\mathcal{L}_B + J_B(x)\phi_B(x))\ ,\right\} \tag{9.33}$$

where we put $\hbar = 1$, while $Z[J]$ is the generating functional when the Lagrangian is $\mathcal{L} + \delta\mathcal{L}$, and the source is J and the functional integration variable is $\phi(x)$:

$$Z[J] = Z_B[J_B] = Z_B[Z_\phi^{-\frac{1}{2}}J]\,, \tag{9.34}$$

which follows from (9.31) and (9.32). This relation can be used to relate the Green's functions of the bare and renormalized theories. Then using eq. (9.34) and

$$G^{(n)}(x_1,\,\dots,x_n) = \frac{1}{(i)^n}\left.\frac{\delta^n Z[J]}{\delta J(x_1)\dots J(x_n)}\right|_{J=0}\,, \tag{9.35}$$

we find

$$G^{(n)}(x_1,\,\dots,x_n) = \left(Z_\phi^{-\frac{1}{2}}\right)^n G_B^{(n)}(x_1,\,\dots,x_n)\,, \tag{9.36}$$

$$\tilde{G}^{(n)}(p_1,\,\dots,p_n) = \left(Z_\phi^{-\frac{1}{2}}\right)^n \tilde{G}_B^{(n)}(p_1,\,\dots,p_n)\,. \tag{9.37}$$

Let us show it for (9.36).

$$\begin{aligned} G^{(n)}(x_1,\,\dots,x_n) &= \frac{1}{(i)^n}\left.\frac{\delta^n Z[J]}{\delta J(x_1)\dots J(x_n)}\right|_{J=0} = \frac{1}{(i)^n}\left.\frac{\delta^n Z_B[J_B]}{\delta J(x_1)\dots J(x_n)}\right|_{J=0} \\ &= \frac{1}{(i)^n}\underbrace{\frac{\delta J_B(x_1)\dots J_B(x_n)}{\delta J(x_1)\dots J(x_n)}}_{(Z_\phi^{-1/2})^n}\left.\frac{\delta^n Z_B[J_B]}{\delta J_B(x_1)\dots J_B(x_n)}\right|_{J_B=0} = \left(Z_\phi^{-\frac{1}{2}}\right)^n G_B^{(n)}(x_1,\,\dots,x_n)\,. \end{aligned} \tag{9.38}$$

We can do the same with the generating functionals of the vertex functions. From (9.34) we obtain that

$$W[J] = W_B[J_B]\,, \tag{9.39}$$

so that

$$\phi_{\rm cl}(x) = \frac{\delta W[J]}{\delta J(x)} = Z_\phi^{-\frac{1}{2}}\frac{\delta W_B}{\delta J_B(x)} = Z_\phi^{-\frac{1}{2}}\phi_{\rm cl}^B\,, \tag{9.40}$$

and hence by recalling $\Gamma = W - \int d^4x J\phi$, we have

$$\Gamma[\phi_{\rm cl}] = \Gamma_B[\phi_{\rm cl}^B] = \Gamma_B[Z_\phi^{\frac{1}{2}}\phi_{\rm cl}]\,. \tag{9.41}$$

Then since we can write

$$\begin{aligned} \Gamma^{(n)}(x_1,\,\dots,x_n) &= \frac{\delta\Gamma(x_1,\dots,x_n)}{\delta\phi_{\rm cl}(x_1)\dots\delta\phi_{\rm cl}(x_n)} = \frac{\delta\Gamma_B(x_1,\dots,x_n)}{\delta\phi_{\rm cl}(x_1)\dots\delta\phi_{\rm cl}(x_n)} \\ &= \underbrace{\frac{\delta\phi_{\rm cl}^B(x)}{\delta\phi_{\rm cl}(x)}}_{(Z_\phi^{1/2})^n}\frac{\delta\Gamma_B(x_1,\dots,x_n)}{\delta\phi_{\rm cl}^B(x_1)\dots\delta\phi_{\rm cl}^B(x_n)}\,, \end{aligned} \tag{9.42}$$

then we can write

$$\Gamma^{(n)}(x_1,\, \dots, x_n) = \left(Z_\phi^{\frac{1}{2}}\right)^n \Gamma_B^{(n)}(x_1,\, \dots, x_n)\,,$$
$$\tilde{\Gamma}^{(n)}(p_1,\, \dots, p_n) = \left(Z_\phi^{\frac{1}{2}}\right)^n \tilde{\Gamma}_B^{(n)}(p_1,\, \dots, p_n)\,. \tag{9.43}$$

For $n = 2$, using eq. (7.37) and (7.39), we have

$$\begin{aligned}\tilde{\Gamma}^{(2)}(p) &= Z_\phi \tilde{\Gamma}_B^{(2)}(p)\\ &= Z_\phi \left\{p^2 - m_B^2\left(1 - \frac{\lambda_B}{16\pi^2}\frac{1}{\varepsilon}\right) + \frac{m_B^2\lambda_B}{32\pi^2}\left[1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_B^2}\right)\right] + \mathcal{O}(\lambda_B^2)\right\}\end{aligned} \tag{9.44}$$

and this last quantity coincides with (9.25) using

$$\begin{cases} Z_\phi m_B^2 = m^2 + \delta m_1^2\,,\\ Z_\phi^2 \lambda_B \mu^\varepsilon = \lambda\mu^\varepsilon + \delta\lambda_2\,. \end{cases} \tag{9.45}$$

Indeed with $Z_\phi = 1 + \delta Z_1$ with δZ_1 of order λ , and $\lambda_B = \lambda$ at order λ (the coupling constant starts to have corrections at order λ^2) we can rewrite (9.44) as follows

$$\tilde{\Gamma}^{(2)}(p) = p^2(1 + \delta Z_1) - m^2 - \delta m_1^2 + \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} + \frac{\lambda m^2}{32\pi^2}\left[1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]\,. \tag{9.46}$$

The last equation coincides with (9.25). So we see that the divergence in $\tilde{\Gamma}^{(2)}$ has been reabsorbed into the renormalization constants and the same is true for $\tilde{\Gamma}^{(4)}$. We have in this way verified that $\lambda\phi^4$ is renormalizable at one loop. The proof that is renormalizable at all orders is more difficult.

10 Renormalization schemes

The precise way in which the parameters $\delta\lambda, \delta m^2$ and δZ_ϕ of the counter term Lagrangian are fixed is called a "*Renormalization Scheme*". The simplest scheme particularly suited to gauge theories is the one that emerges naturally from dimensional regularization (DR): *the minimal subtraction scheme*, MS scheme. In such scheme the counter terms remove only the divergence. Let us see the details.
In DR the Green's functions develop poles and higher order singularities in $D-4$. In any renormalization scheme those singularities are removed by the counterterms, which have the general form:

$$\delta\lambda = \mu^\varepsilon \left(a_0 \left(\lambda, \frac{\mu}{m}, D \right) + \sum_{v=1}^{\infty} \frac{a_v(\lambda, \frac{\mu}{m})}{\varepsilon^v} \right) , \tag{10.1}$$

$$\delta m^2 = m^2 \left(b_0 \left(\lambda, \frac{\mu}{m}, D \right) + \sum_{v=1}^{\infty} \frac{b_v(\lambda, \frac{\mu}{m})}{\varepsilon^v} \right) , \tag{10.2}$$

$$\delta Z_\phi = c_0 \left(\lambda, \frac{\mu}{m}, D \right) + \sum_{v=1}^{\infty} \frac{c_v(\lambda, \frac{\mu}{m})}{\varepsilon^v} , \tag{10.3}$$

where λ is the renormalized dimensionless coupling constant, and the counterterms are expressed as Laurent series in terms of the renormalized parameters. We notice that

- a_0, b_0, c_0 are regular for $\varepsilon \to 0$;
- a_v, b_v, c_v are dimensionless: they can depend only on λ and μ/m.

In MS scheme these counterterms remove only the singularities in ε so that we have:

$$a_0^{\rm MS} = b_0^{\rm MS} = c_0^{\rm MS} = 0 . \tag{10.4}$$

The very nice characteristic of MS scheme is that the other coefficients a_v, b_v, c_v turn out to be mass independent. Hence, for example, we have

$$a_v^{\rm MS} \left(\lambda, \frac{\mu}{m} \right) \equiv a_v(\lambda) . \tag{10.5}$$

We can verify this using the results we have obtained up to now. Let us compare eq (9.30):

$$\frac{3\lambda^2 \mu^\varepsilon}{16\pi^2 \varepsilon} - \delta\lambda_2 \to \text{ constant} , \tag{10.6}$$

with the equation (10.1) and use (10.4), we find

$$\delta\lambda^{\rm MS} = \frac{3\lambda^2 \mu^\varepsilon}{16\pi^2 \varepsilon} + \mathcal{O}(\lambda^3) , \tag{10.7}$$

and so we obtain

$$a_1^{\rm MS} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) , \tag{10.8}$$

so that we can see that at this order $a_1^{\rm MS}$ is mass independent as it is claimed. In the same way we can infer from eq. (9.25), that we repeat here,

$$\tilde{\Gamma}^{(2)}(p) = p^2(1 + \delta Z_1) - m^2 - \delta m_1^2 + \frac{\lambda m^2}{16\pi^2} \frac{1}{\varepsilon} + \frac{\lambda m^2}{32\pi^2} \left[1 - \gamma_E + \ln \left(\frac{4\pi\mu^2}{m^2} \right) \right] , \tag{10.9}$$

the following relations for the b_1 and c_1 coefficients

$$\delta m^2_{\rm MS} = \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} + \mathcal{O}(\lambda^2) \ \Rightarrow\ b_1^{\rm MS} = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)\,,$$
$$\delta Z_\phi^{\rm MS} = \mathcal{O}(\lambda^2) \ \Rightarrow\ c_1^{\rm MS} = \mathcal{O}(\lambda^2)\,. \tag{10.10}$$

The mass independence of the counterterms is precious when one studies RG equations as we will see.
Now that we have fixed the counterterms the renormalized Green's functions are finite and unambiguous functions of the renormalized parameters and we can take the limit $\varepsilon \to 0$. In MS we have:

$$\tilde\Gamma^{(2)}(p) = p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\left(1-\gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right) + \mathcal{O}(\lambda^2)\,, \tag{10.11}$$
$$\tilde\Gamma^4(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left\{3\gamma_E + 3\ln\frac{m^2}{4\pi\mu^2} - A(s,m) - A(t,m) - A(u,m)\right\}$$
$$+\,\mathcal{O}(\lambda^3)\,, \tag{10.12}$$

with:

$$A(s,m) \equiv 2 - 2\left(\frac{4m^2}{s}-1\right)^{\frac12}\tan^{-1}\left(\frac{4m^2}{s}-1\right)^{-\frac12}. \tag{10.13}$$

To derive eq. (10.12) we used eq. (9.28), with

$$F(s,m,\mu) = \int_0^1 dx\,\ln\left(\frac{m^2 - x(1-x)s}{4\pi^2\mu^2}\right) = \ln\frac{m^2}{4\pi\mu^2} - A(s,m)\,, \tag{10.14}$$

where

$$\begin{aligned} A(s,m) &= -\int_0^1 dx\,\ln\left(1 - \frac{s}{m^2}x(1-x)\right) \\ &= -\int_0^1 dx\,\ln\left(1 - \frac{s}{4m^2} + \frac{s}{m^2}(x-\frac12)^2\right) \\ &= 2 - 2\left(\frac{4m^2}{s}-1\right)^{\frac12}\tan^{-1}\left(\frac{4m^2}{s}-1\right)^{-\frac12}. \end{aligned} \tag{10.15}$$

Note that in (10.11) and (10.12) there is an implicit dependence of the parameter λ and m^2 on the renormalization scheme.
The MS scheme is just one scheme in a class of mass-independent schemes: in all of them a_v, b_v, c_v are independent of μ/m but in the others a_0 is not necessarily equal to zero.
Indeed the constant $-\gamma_E + \ln 4\pi$ comes from the expansion around $D = 4-\varepsilon$ of the integrals evaluated in DR and appears in that combination.

In the $\overline{\rm MS}$ scheme those (finite and mass independent) constants are also subtracted by the counter terms. So we have:

$$\delta\lambda^{\overline{\rm MS}} = \frac{3\lambda^2\mu^\varepsilon}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi\right) + \mathcal{O}(\lambda^3)\,, \tag{10.16}$$

and hence for the coefficients a_0 and a_1:

$$a_1^{\overline{\rm MS}} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)\,, \tag{10.17}$$

$$a_0^{\overline{\rm MS}} = \frac{3\lambda^2}{32\pi^2}\left(-\gamma_E + \ln 4\pi\right) + \mathcal{O}(\lambda^3)\,. \tag{10.18}$$

For the coefficients b_0 and b_1 we have to look at

$$\delta m^{2\,\rm MS} = \frac{\lambda\, m^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi\right) + \mathcal{O}(\lambda^2)\,, \tag{10.19}$$

and then we obtain

$$b_1^{\overline{\rm MS}} = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)\,, \tag{10.20}$$

$$b_0^{\overline{\rm MS}} = \frac{m^2\,\lambda}{32\pi}\left(-\gamma_E + \ln 4\pi\right) + \mathcal{O}(\lambda^2)\,. \tag{10.21}$$

We then find in $\overline{\rm MS}$ scheme the following expressions for the vertex functions:

$$\tilde{\Gamma}^2(p) = p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\left(1 + \ln\frac{\mu^2}{m^2}\right) + \mathcal{O}(\lambda^2)\,, \tag{10.22}$$

$$\tilde{\Gamma}^4(p_1,p_2,p_3,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left\{3\ln\frac{m^2}{\mu^2} - A(s,m^2) - A(t,m^2) - A(u,m^2)\right\}\,. \tag{10.23}$$

In other two important schemes that we are going to discuss, the counterterms are determined imposing boundary conditions on $\tilde{\Gamma}^2$ and $\tilde{\Gamma}^4$ (*renormalization conditions)*.
Three conditions are needed to fix the three quantities $\delta\lambda, \delta m^2, \delta Z_\phi$.

10.1 Momentum scheme (MOM)

This is used and introduced for theories with zero mass such that we cannot impose renormalization conditions of $p^2 = m^2$ because of $p^2 = 0$. When there is a massless particle, the Green functions typically have infrared divergences. For on-shell particles we have $p^2 = m^2$, and consequently

$$s + t + u = 4m^2\,. \tag{10.24}$$

Here we impose the boundary condition at a point in momentum space which does not correspond to the momentum of a physical particle. The boundary conditions are imposed at a point where the external momenta are characterized by a single scale M.
The conditions are such that at the (Euclidean) point $p^2 = -M^2$:

$$\tilde{\Gamma}^2(p)\Big|_{p^2=-M^2} = -m^2 - M^2\,, \tag{10.25}$$

$$\frac{\partial\tilde{\Gamma}^2(p)}{\partial p^2}\Big|_{p^2=-M^2} = 1\,. \tag{10.26}$$

We need two conditions to fix the mass and the field renormalization. These conditions require that:

$$\tilde{\Gamma}^2(p) = p^2 - m^2 + \mathcal{O}[(p^2+M^2)^2]^2\,. \tag{10.27}$$

If we go back to our eq. for $\tilde{\Gamma}^2(p)$ in terms of external momenta, renormalized parameters and counterterms given in (9.25)

$$\tilde{\Gamma}^{(2)}(p) = p^2(1+\delta Z_1) - m^2 - \delta m_1^2 + \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} + \frac{\lambda m^2}{32\pi^2}\left[1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right], \qquad (10.28)$$

we see that (10.27) requires

$$\delta Z^{\mathrm{MOM}} = \mathcal{O}(\lambda^2), \qquad (10.29)$$

$$\delta m_1^{2,MOM} = \frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\varepsilon} + 1 - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right) + \mathcal{O}(\lambda^2). \qquad (10.30)$$

We notice that $\delta\lambda$ is fixed by a similar boundary condition on $\tilde{\Gamma}^4$:

$$\tilde{\Gamma}^4(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j = M^2\left(\frac{1}{3}-\frac{4}{3}\delta_{ij}\right)} = -\lambda, \quad i,j = 1,...,4, \qquad (10.31)$$

and the symmetric case for $i = j$ is

$$s = t = u = -\frac{4}{3}M^2. \qquad (10.32)$$

Note that with dimensional regularization the condition $\tilde{\Gamma}^4$ can be written as

$$\tilde{\Gamma}^4(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j = M^2\left(\frac{1}{3}-\frac{4}{3}\delta_{ij}\right)} = -\lambda\mu^\varepsilon, \quad i,j = 1,...,4, \qquad (10.33)$$

so that λ remains dimensionless. From our previous equation on the renormalized $\tilde{\Gamma}^4$ in eq. (9.28):

$$\begin{aligned}\tilde{\Gamma}^4(p_1,p_2,p_3,p_4) &= -\lambda\mu^\varepsilon + \frac{3\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon} - \frac{\lambda^2\mu^\varepsilon}{32\pi^2}\left\{3\gamma_E + 3\ln\frac{m^2}{4\pi\mu^2} - A(s,m) - A(t,m) - A(u,m)\right\} \\ &\quad -\delta\lambda_2 + \mathcal{O}(\lambda^3),\end{aligned} \qquad (10.34)$$

so that

$$\delta\lambda_2^{\mathrm{MOM}} = \frac{\lambda^2\mu^\varepsilon}{32\pi^2}\left(\frac{6}{\varepsilon} - 3\gamma_E - 3\ln\frac{m^2}{4\pi\mu^2} + 3A\left(-\frac{4}{3}M^2, m^2\right)\right) + \mathcal{O}(\lambda^3). \qquad (10.35)$$

Therefore the renormalized proper Green's functions in MOM scheme, as $\varepsilon \to 0$, are:

$$\tilde{\Gamma}^2(p) = p^2 - m^2 + \mathcal{O}(\lambda^2), \qquad (10.36)$$

$$\begin{aligned}\tilde{\Gamma}^4(p_1,p_2,p_3,p_4) &= -\lambda + \frac{\lambda^2}{32\pi^2}\left[A(s,m^2) + A(t,m^2) + A(u,m^2) - 3A\left(-\frac{4}{3}M^2, m^2\right)\right] \\ &\quad + \mathcal{O}(\lambda^3).\end{aligned} \qquad (10.37)$$

In this scheme the mass scale μ has disappeared from the Green's functions and it has been replaced by the scale M, but the counterterms $\delta m^2, \delta\lambda$ are manifestly NOT mass-independent. (the numerical value $\ln(M^2/m^2)$ and $\ln(M^2/m_B^2)$ becomes relevant in choosing M).

Notice: in all the schemes that we have discussed up to now the quantities m^2 and λ that are called the "renormalized mass" and the "renormalized coupling constant" are only the parameters with which we choose to characterize the Green's functions. Instead in the "on-shell" or "physical schemes" m and λ are the mass and the coupling constant, hence they are the values that are actually measured.

On shell scheme

The physical mass is defined as the position of the pole $\tilde{G}^{(2)}$ or equivalently as the position of the zero in $\tilde{\Gamma}^2$. So in this scheme the renormalization conditions are:

$$\tilde{\Gamma}^{(2)}(p)\Big|_{p^2=m^2} = 0\,, \tag{10.38}$$

$$\frac{\partial\tilde{\Gamma}^2}{\partial p^2}(p)\Big|_{p^2=m^2} = 1\,, \tag{10.39}$$

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j=-m^2\left(\frac{1}{3}-\frac{4}{3}\delta_{ij}\right)} = -\lambda\,. \tag{10.40}$$

Similar to the MOM scheme, with dimensional regularization the condition $\tilde{\Gamma}^4$ can be written as

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j=-m^2\left(\frac{1}{3}-\frac{4}{3}\delta_{ij}\right)} = -\lambda\mu^{\varepsilon}\,, \tag{10.41}$$

so that λ remains dimensionless. Comparing these conditions with the analogous conditions in the MOM scheme

$$\tilde{\Gamma}^{(2)}(p)\Big|_{p^2=-M^2} = 0\,, \tag{10.42}$$

$$\frac{\partial\tilde{\Gamma}^2}{\partial p^2}(p)\Big|_{p^2=-M^2} = 0\,, \tag{10.43}$$

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4)\Big|_{p_i\cdot p_j=M^2\left(\frac{1}{3}-\frac{4}{3}\delta_{ij}\right)} = -\lambda\,. \tag{10.44}$$

We see that the on shell scheme is only a special case of the momentum scheme in which $M^2 = -m^2$. Therefore the renormalized Green's functions, as $\varepsilon \to 0$, are:

$$\tilde{\Gamma}^2(p) = p^2 - m^2 + \mathcal{O}(\lambda^2)\ , \tag{10.45}$$

$$\tilde{\Gamma}^4(p_1,p_2,p_3,p_4) = -\lambda + \frac{\lambda^2}{32\pi^2}[A(s,m^2)+A(t,m^2)+A(u,m^2)-3A(\frac{4}{3}m^2,m^2)] + \mathcal{O}(\lambda^3)\,, \tag{10.46}$$

and the counter terms are:

$$\delta Z_\phi^{\text{on shell}} = \mathcal{O}(\lambda^2)\,, \tag{10.47}$$

$$\delta m^{2,\text{on shell}} = \frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\varepsilon}+1-\gamma_E+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right)\,, \tag{10.48}$$

$$\delta\lambda^{\text{on shell}} = \frac{\lambda^2\mu^\varepsilon}{32\pi^2}\left(\frac{6}{\varepsilon}-3\gamma_E-3\ln\frac{m^2}{4\pi\mu^2}+3A\left(\frac{4}{3}m^2,m^2\right)\right)+\mathcal{O}(\lambda^3)\,. \tag{10.49}$$

Notice: the only mass parameter appearing in the Green's functions is m. However the counterterms remain mass dependent.

Finally we remark that any Green's functions has a unique value independent of the scheme used to define the parameters. This means that *the values of the parameters differ by finite amounts which depend upon the scheme adopted.*
For example comparing the form of $\tilde{\Gamma}^2$ and $\tilde{\Gamma}^4$ in $\overline{\text{MS}}$ and MOM scheme:

$$\tilde{\Gamma}^{(2)}_{\overline{\text{MS}}}(p) = p^2 - m^2_{\overline{\text{MS}}} - \frac{\lambda_{\overline{\text{MS}}}\, m^2_{\overline{\text{MS}}}}{32\pi^2}\left(1+\ln\frac{\mu^2}{m^2_{\overline{\text{MS}}}}\right)+\mathcal{O}(\lambda^2_{\overline{\text{MS}}})\,, \tag{10.50}$$

$$\tilde{\Gamma}^{(4)}_{\overline{\rm MS}}(p_1,p_2,p_3,p_4) = -\lambda_{\overline{\rm MS}} - \frac{\lambda^2_{\overline{\rm MS}}}{32\pi^2}\left\{3\ln\frac{m^2_{\overline{\rm MS}}}{\mu^2} - A(s,m^2_{\overline{\rm MS}}) - A(t,m^2_{\overline{\rm MS}}) - A(u,m^2_{\overline{\rm MS}})\right\}\,, \tag{10.51}$$

vs

$$\tilde{\Gamma}^{(2)}_{\rm MOM}(p) = p^2 - m^2_{\rm MOM} + \mathcal{O}(\lambda^2_{\rm MOM})\,, \tag{10.52}$$

$$\begin{aligned}\tilde{\Gamma}^{(4)}_{\rm MOM}(p_1,p_2,p_3,p_4) = -\lambda_{\rm MOM} + \frac{\lambda^2_{\rm MOM}}{32\pi^2}\Big[A(s,m^2_{\rm MOM}) + A(t,m^2_{\rm MOM})& \\ + A(n,m^2_{\rm MOM}) - 3A\left(-\frac{4}{3}M^2, m^2_{\rm MOM}\right)\Big] + \mathcal{O}(\lambda^3_{\rm MOM})\,,&\end{aligned} \tag{10.53}$$

we find

$$m^2_{\rm MOM} = m^2_{\overline{\rm MS}}\left[1 - \frac{\lambda_{\overline{\rm MS}}}{32\pi^2}\left(1 + \ln\frac{\mu^2}{m^2_{\overline{\rm MS}}}\right)\right] + \mathcal{O}(\lambda^2_{\overline{\rm MS}})\,, \tag{10.54}$$

$$\lambda_{\rm MOM} = \lambda_{\overline{\rm MS}} + \frac{3\lambda^2_{\overline{\rm MS}}}{32\pi^2}\left(\ln\frac{m^2_{\overline{\rm MS}}}{\mu^2} - A\left(-\frac{4}{3}M^2, m^2_{\overline{\rm MS}}\right)\right) + \mathcal{O}(\lambda^3_{\overline{\rm MS}})\,, \tag{10.55}$$

where we have used

$$\begin{cases} m^2_{\rm MOM} = m^2_{\overline{\rm MS}} + \mathcal{O}(\lambda)\,, \\ \lambda_{\rm MOM} = \lambda_{\overline{\rm MS}} + \mathcal{O}(\lambda^2)\,. \end{cases} \tag{10.56}$$

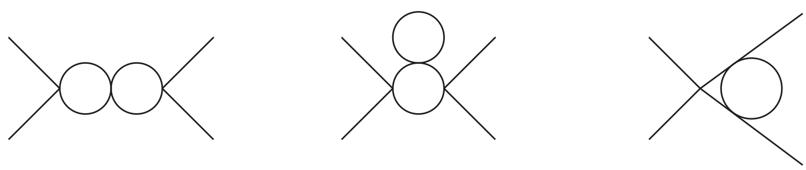
Notice: the renormalization conditions are arbitrary, hence if we change the renormalization conditions, the Green's functions must change in a definite way, so that the observable quantities are invariant under this transformation $\to$ the property is the invariance under the Renormalization Group (RG).

10.2 Renormalization of $\lambda\phi^4$ beyond one loop

We have now renormalized $\lambda\phi^4$ to one loop. What happens at 2 loops? The relevant diagrams are:

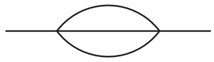
$\tilde{\Gamma}^{(2)}$:

$\tilde{\Gamma}^{(4)}$:



It is found that for $\tilde{\Gamma}^{(2)}$ the addition of two loop diagrams changes the renormalized mass m^2 (as we would expect δm^2 is a series in λ and gets new contribution at each new loop) but $\tilde{\Gamma}^2$ contains also an additional divergence coming from the second diagram in figure. For what concerns $\Gamma^{(4)}$ the 2-loop graphs give extra divergent contributions $\to$ some of

them disappear when the mass renormalization is taken into account and the rest con be made finite by redefining the renormalized coupling constant λ. So $\tilde{\Gamma}^{(4)}$ is finite at two loop with mass and coupling constant renormalization at two loops. To make $\tilde{\Gamma}^{(2)}$ finite at two loop one has to use also δZ that has been found to be zero at one loop. So at two loops the $\tilde{\Gamma}^{(n)}$ functions can be made finite by suitable renormalization.
The question is: does this property holds at any order? This has been proved explicitly in the literature and the prove is difficult.
The biggest difficulty comes from the overlapping divergences. They are present for example in



$$\propto \int \frac{d^4k_1\, d^4k_2}{k_1^2 k_2^2 (p-k_1-k_2)^2}\,, \tag{10.57}$$

where the superficial degree of divergence is $D = 8 - 6 = 2$.
However if k_1 is hold constant and we integrate over k_2, the degree of divergence is $4 - 4 = 0$, so the graph still gives a logarithmic divergence. This is the phenomenon of overlapping divergences $\rightarrow$ it is not possible to separate divergences in k_1 and k_2.

Appendices

A Brief reminder on functional derivatives

A wide range of physics can be formulated in terms of so called variational calculus. The first ingredient is a functional which is a map from a certain space of functions to numbers. Precisely, a functional $F[\phi]$ is a mapping from a normed linear space of functions (a Banach space) $M \equiv \{\phi(x) : x \in \mathbb{R}\}$ to the field of real or complex numbers, $F : M \to \mathbb{R}$ or $\mathbb{C}$.

Example of functionals used in QFT include the action functionals:

$$S[q] \quad = \int_{t_{in}}^{t_{fin}} dt L(q, \dot{q}) \tag{A.1}$$

$$S[\phi] \quad = \int dt L(\phi, \partial_\mu \phi) = \int dt d^3x \mathcal{L}(\phi, \partial_\mu \phi) \tag{A.2}$$

where the first line refers to the action of classical mechanics and the second one to the action of field theory. Another example is the class of generating functionals.

The discussion of field theories makes ample use of functional derivatives, i.e. the differentiation of a functional with respect to its argument. Here we give a simple introduction and give some properties of this mathematical operation. If you want to obtain more precise definitions and details I suggest you to look into mathematical books, like for example Gelfand and Fomin, "Calculus of variations" (Dover Books on Mathematics, 2000). or the volume 1 of Courant and Hilbert, "Methods of Mathematical Physics" (Wiley, 1989).

The object $\delta F[\phi]/\delta\phi(x)$ tells how the value of the functional changes if the function $\phi(x)$ is changed at the pont x. We can define the functional differential (or variation) as

$$\delta F[\phi] = \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \delta \phi(x) \tag{A.3}$$

which tells that the total change in F upon variation of the function $\phi(x)$ is a linear superposition of the local changes summed over the whole range of x values. As in ordinary differentiation the functional derivative can be represented as the limit of divided differences. To see this in concrete we construct a specific variation of the independent variable, i.e. the function $\phi(x)$, which is localized at the point y with strength ε:

$$\delta\phi(x) = \varepsilon \delta(x - y). \tag{A.4}$$

Inserting this into $(A.3)$ we have

$$\delta F[\phi] = F[\phi + \varepsilon\delta(x - y)] - F[\phi] = \int dx \frac{\delta F[\phi(x)]}{\delta\phi(x)} \varepsilon \delta(x - y) = \varepsilon \frac{\delta F}{\delta \phi(y)} \tag{A.5}$$

or, in the limit of vanishing ε

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\delta F[\phi(x)]}{\delta\phi(y)} = \lim_{\varepsilon \to 0} \frac{F[\phi(x) + \varepsilon\delta(x - y)] - F[\phi(x)]}{\varepsilon}. \tag{A.6}$$

Most of the rules of ordinary differential calculus can also apply to functional derivatives. We deal with a linear operation. Therefore given two functionals F and G and two constants λ, μ so the functional derivative satisfies

$$\frac{\delta(\lambda F + \mu G)}{\delta\phi(x)} = \lambda\frac{\delta F}{\delta\phi(x)} + \mu\frac{\delta G}{\delta\phi(x)}. \tag{A.7}$$

The derivative of the combined functional $F[\phi] = G[\phi]H[\phi]$ is given by

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] + G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)}. \tag{A.8}$$

Similarly the chain rule can be applied to the functional of a functional

$$\frac{\delta}{\delta\phi(y)}F[G[\phi]] = \int dx \frac{\delta F[G]}{\delta G(x)}\frac{\delta G[\phi]}{\delta\phi(y)}. \tag{A.9}$$

Then for a functional $F[\phi]$ given by

$$F[\phi] = \int d^4x_1 \ldots d^4x_n f(x_1, \ldots, x_n)\phi(x_1)\ldots\phi(x_n) \tag{A.10}$$

with f symmetric in all variables, we have:

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \int d^4x_1 \ldots d^4x_{n-1}\phi(x_1)\ldots\phi(x_{n-1})\, n\, f(x_1, \ldots, x_{n-1}, x)\,. \tag{A.11}$$

If the functional is given by the series:

$$F[\phi] = \sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1 \ldots d^4x_n f_n(x_1, \ldots, x_n)\phi(x_1)\ldots\phi(x_n)\,, \tag{A.12}$$

we have

$$f_n(x_1, \ldots, x_n) = \left.\frac{\delta^n F[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}\right|_{J=0}\,. \tag{A.13}$$

B Saddle Point Approximation

Consider the integral

$$I = \int_{-\infty}^{+\infty} dq\, \mathrm{e}^{-f(q)/\hbar}. \tag{B.1}$$

For very small values of $\hbar$ (as $\hbar \to 0$), the integral is dominated by the minimum of $f(q)$ (for simplicity we assume that the integral has only one minimum). Using the power expansion of $f(x)$ about its minimum, denoted by q_0,

$$f(q) = f(q_0) + \frac{1}{2}f''(q_0)\,(q - q_0)^2 + \mathcal{O}(q - q_0)^3\,, \tag{B.2}$$

one can simplify the integral as

$$I = \mathrm{e}^{-f(q_0)/\hbar}\int_{-\infty}^{+\infty} dq\, \mathrm{e}^{-\frac{1}{2\hbar}f''(q_0)\,(q-q_0)^2+\mathcal{O}(q-q_0)^3}\,,$$

$$\rightarrow \quad \mathrm{e}^{-f(q_0)/\hbar} \int_{-\infty}^{+\infty} dq \; \mathrm{e}^{-\frac{1}{2\hbar} f''(q_0)\,(q-q_0)^2} \,. \tag{B.3}$$

It is straightforward to calculate the Gaussian integral in the above expression. Using the change of variable $x = (q - q_0)/\sqrt{\hbar}$, one can easily find

$$I = \mathrm{e}^{-f(q_0)/\hbar} \sqrt{\frac{2\pi\hbar}{f''(q_0)}} \; \mathrm{e}^{-\mathcal{O}(\hbar^{1/2})} \,. \tag{B.4}$$

In the case of the action $S\,[x]$, which appears in the path integral calculations, one needs to use the functional derivatives. Similar to the above example, one can see that, as $\hbar \to 0$, the path integral is dominated by the classical solution

$$\frac{\delta S}{\delta x} = 0 \,, \tag{B.5}$$

with appropriate boundary condition. Therefor, for $\hbar \to 0$ only the classical solution remains.

C Wick rotation

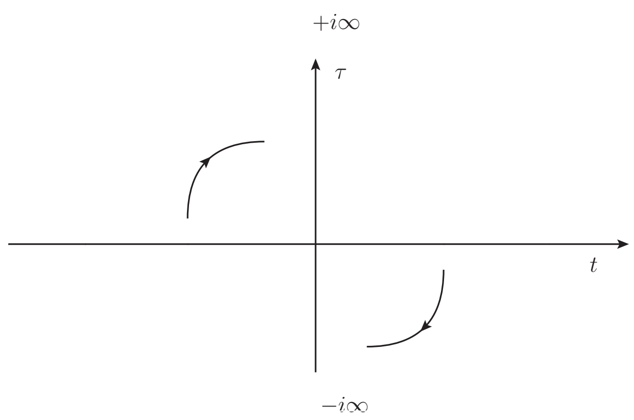
We have seen in the path integral formulation of QFT that might be useful to slightly tip into the complex plane the time integration. This corresponds to an analytic continuation from real to imaginary times. One can think of the rotation for a momentum integral (and then changing the energy of the particle into a complex energy) or for the time coordinate. In the latter case we have

$$t \rightarrow i\tau \,, \tag{C.1}$$

that can be seen also as a rotation of the temporal axis

$$t \rightarrow e^{-i\theta} t = \tau \Rightarrow -it = \tau \Rightarrow t = i\tau \,, \tag{C.2}$$

and for $\theta = \pi/2$ we have the Wick rotation.



D Gaussian integration: analytic continuation

We have defined functional integrals as the appropriate limit of multiple integrals. One typically encounters integrals of the type:

$$I = \int_{-\infty}^{+\infty} dx\, exp(-b\epsilon x^2 + ibx^2) = 2\int_0^{+\infty} dx\, exp(-bx^2(\epsilon - i))\,, \tag{D.1}$$

with $\epsilon \to 0$ that corresponds to the $(+i\epsilon)$ prescription.
Rotating the contour of integration $x' = xe^{i\varphi}$ so that $e^{-2i\varphi}(\epsilon - i) = 1$ one gets

$$I = \frac{1}{\sqrt{\epsilon - i}} \left(\frac{\pi}{b}\right)^{\frac{1}{2}} \underset{\epsilon\to 0}{\to} \left(\frac{i\pi}{b}\right)^{\frac{1}{2}}\,, \tag{D.2}$$

which is the analytic continuation of the gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \tag{D.3}$$

for complex a ($\mathrm{Re}\, a > 0$).
A similar result holds for integration over the complex variable $z = x + iy$.

Gaussian integrals

From the known equation:

$$\int_{-\infty}^{+\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}\,, \tag{D.4}$$

we can simply obtain

$$\int_{-\infty}^{+\infty} dx_1 ... dx_n e^{-\frac{1}{2}\sum_{k=1}^n a_k x_k^2} = \prod_{k=1}^n \left(\frac{2\pi}{a_k}\right)^{\frac{1}{2}}\,. \tag{D.5}$$

We can write this result using scalar products and matrices. We consider $x, y \in \mathbb{R}^n$ so that

$$x \cdot y = (x, y) = \sum_{k=1}^n x_k y_k\,, \tag{D.6}$$

and we consider a diagonal matrix

$$A = \begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{pmatrix} \tag{D.7}$$

so that

$$(x, Ax) = \sum_{k=1}^n a_k x_k^2\,. \tag{D.8}$$

One can also define the integral measure as

$$[dx] = \frac{dx_1 ... dx_n}{(2\pi)^{\frac{n}{2}}} = \frac{dx^n}{(2\pi)^{\frac{n}{2}}}\,. \tag{D.9}$$

So that we can finally write

$$\int [dx]\, exp\left[-\frac{1}{2}(x, Ax)\right] = \frac{1}{\sqrt{\det A}}\,. \tag{D.10}$$

If A is not a diagonal matrix but it is symmetric, then it can always be diagonalized by an orthogonal transformation:

$$O^{-1}AO = diag(a_1...a_n)\,,\ \text{with}\ O^{-1} = O^T, \quad O \in O(n)\,. \tag{D.11}$$

Then, one can perform on the integral the transformation $x \to Ox$, $dx = dOx = dx$ because the determinant of O is 1 (being an orthogonal matrix). Then we write

$$\begin{aligned} \int [dx]\, exp\left\{-\frac{1}{2}(x, Ax)\right\} &= \int [dOx] exp\left\{-\frac{1}{2}(Ox, AOx)\right\} \\ &= \int [dx]\, exp\left\{-\frac{1}{2}(x, diag(a_1...a_n)x)\right\} \\ &= \frac{1}{\sqrt{detA}} \end{aligned} \tag{D.12}$$

where the exponent has been simplified as follows

$$(Ox, AOx) = (x, O^T AOx) = (x, diag(a_1...a_n)x)\,. \tag{D.13}$$

Quadratical forms

We can start with the quadratic form

$$q(x) = -\frac{a}{2}x^2 + bx + c = q(x_0) - \frac{a}{2}(x - x_0)^2\,, \tag{D.14}$$

with $x_0 = b/a$ and

$$\begin{cases} \frac{\partial q}{\partial x} = -ax + b, \quad \left.\frac{\partial q}{\partial x}\right|_{x=x_0} = 0\,, \\ q(x_0) = -a\frac{b^2}{2a^2} + \frac{b^2}{a} + c = \frac{b^2}{2a} + c\,. \end{cases} \tag{D.15}$$

Then we have

$$\int_{-\infty}^{+\infty} dx\, exp(-\frac{a}{2}x^2 + bx + c) = exp(q(x_0))\sqrt{\frac{2\pi}{a}} = exp\left(\frac{b^2}{2a} + c\right)\sqrt{\frac{2\pi}{a}}\,, \tag{D.16}$$

and we can generalize this formula to vectors $x \in \mathbb{R}^n$ and matrices $n \times n$. Then, we have

$$Q(x) = -\frac{1}{2}(x, Ax) + (b, x) + c \tag{D.17}$$

with A positive and invertible matrix. We put

$$x_0 = A^{-1}b \tag{D.18}$$

and so

$$\begin{aligned} Q(x) &= Q(x_0) - \frac{1}{2}(x - x_0, A(x - x_0)) \\ &= \frac{1}{2}(b, A^{-1}b) + c - \frac{1}{2}(x - x_0, A(x - x_0)) \end{aligned} \tag{D.19}$$

and we obtain, using (D.16)

$$\int [dx]\, e^{+Q(x)} = exp\left[\frac{1}{2}(b, A^{-1}b) + c\right]\frac{1}{\sqrt{detA}}\,. \tag{D.20}$$

Complex variables and Hermitian matrices

From what we have seen we can calculate

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{-a\frac{(x^2+y^2)}{2}} = \frac{2\pi}{a}\,, \tag{D.21}$$

and we can write $z = \frac{1}{\sqrt{2}}(x+iy)$, $z^* = \frac{1}{\sqrt{2}}(x-iy)$ with $z \in \mathbb{C}$; and $x, y \in \mathbb{R}$. Performing then the change of variables:

$$dxdy = \left|\frac{\partial(x,y)}{\partial(z,z^*)}\right| dzdz^* = \left|\det\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\right| = dz^*dz\,, \tag{D.22}$$

from (D.21) we can obtain

$$\int \frac{dzdz^*}{2\pi} e^{-az^*z} = \frac{1}{a} \tag{D.23}$$

and this can be generalized to

$$\int \frac{dz_1 dz_1^* ... dz_n dz_n^*}{(2\pi)^N} e^{-(z^*, Az)} = (\mathrm{detA})^{-1} = e^{-\mathrm{tr}\ln A} \tag{D.24}$$

for any positive definite matrix A that can be diagonalized by unitary transformation. In the last equality we use a relation which is shown below (see Eq. D.31). To define Gaussian integration over fields we will generalize the results that hold for N-dimensional vector spaces to infinite dimensional functional spaces. For real functions $\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ we define the standard scalar product

$$\begin{aligned} (\phi, \psi) &= \int d^4x \phi(x)\psi(x)\,, \\ A\phi(x) &= \int d^4y A(x,y)\phi(y) \end{aligned} \tag{D.25}$$

Any action quadratic in the fields can be written in general as

$$S = \int d^4x d^4y \phi(x) A(x,y) \phi(y) \tag{D.26}$$

For this set of actions we obtain

$$\int \mathcal{D}\phi \,\exp\left\{-\frac{1}{2}\int d^4x \int d^4y \phi(x)A(x,y)\phi(y)\right\} = \frac{1}{\sqrt{\det A}}\,, \tag{D.27}$$

$$\begin{aligned} &\int \mathcal{D}\phi \,\exp\left\{-\frac{1}{2}\int d^4x \int d^4y \phi(x)A(x,y)\phi(y) + J(x)\phi(x)\right\} \\ &= \frac{1}{\sqrt{\det A}} \exp\left\{\frac{1}{2}\int d^4x \int d^4y J(x)A^{-1}(x,y)J(y)\right\}\,, \end{aligned} \tag{D.28}$$

$$\int \mathcal{D}\phi \int \mathcal{D}\phi^* \,\exp\left\{-\int d^4x \int d^4y \phi^*(x)A(x,y)\phi(y)\right\} = (\det A)^{-1}\,, \tag{D.29}$$

where the 2π factors have been included in the definition of the functional measure (this is not relevant as we will always deal with ratios of such integrals). For any matrix A that can be diagonalized by unitary transformations there are some useful formulas:

$$\begin{cases} \det(1-A) = \exp\{\mathrm{tr}\,\ln(1-A)\}\,, \\ \det A = \exp\{\mathrm{tr}\,\ln A\}\,, \\ \mathrm{tr}\,\ln(1-A) = -\mathrm{tr}[A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + ...]\,. \end{cases} \tag{D.30}$$

with

$$\det A = \Pi_i a_i = \exp\left(\ln \Pi_i a_i\right) = \exp\left(\sum_i \ln a_i\right) = \exp\left(\mathrm{tr} \ln A\right) \ , \tag{D.31}$$

where we have used the fact that the determinant of the matrix A is the product of its eigenvalues a_i.

E D-dimensional Integrals and Gamma Function

In this appendix our focus is on integrals of type

$$\int_{-\infty}^{\infty} d^D p \, f(p^2) \tag{E.1}$$

with an arbitrary dimension D. Since the integrand is only a function of p^2, one can use "spherical" coordinates to simplify the integral as

$$\int_{-\infty}^{\infty} d^D p \, f(p^2) = \int d\Omega_D \int_0^{+\infty} dp \, p^{D-1} f(p^2) \, , \tag{E.2}$$

where

$$\Omega_D = \int d\Omega_D \equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin(\theta_1) d\theta_1 \int_0^{\pi} \sin^2(\theta_2) d\theta_2 \ \cdots \ \int_0^{\pi} \sin^{D-2}(\theta_{D-1}) d\theta_{D-2}. \tag{E.3}$$

Ω_D is the surface area of a D dimensional unit sphere. For instance in $D = 3$ we have

$$\Omega_3 = \int d\Omega_3 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta_1 d\theta_1 = 4\pi \ . \tag{E.4}$$

If we can calculate Ω_D, then the integral in (E.2) is reduced to a one dimensional integral, where its dependence on D is completely well defined when the integral is convergent. To calculate Ω_D for an arbitrary D we can use a D dimensional Gaussian integral and evaluate it in both Cartesian and spherical coordinates. For an integer value of D, using Cartesian coordinates we obtain

$$\int d^D x \exp\Big(-\sum_{i=1}^{D} x_i^2\Big) = \Big[\int dx e^{-x^2}\Big]^D = (\sqrt{\pi})^D \ . \tag{E.5}$$

On the other hand, using spherical coordinates, we have

$$\begin{aligned} \int d^D x \exp\Big(-\sum_{i=1}^{D} x_i^2\Big) &= \int d\Omega_D \int_0^{\infty} dr \, r^{D-1} e^{-r^2} \\ &= \Omega_D \frac{1}{2} \int_0^{\infty} dx \, x^{D/2-1} e^{-x} \\ &= \Omega_D \frac{1}{2} \Gamma(D/2) \, , \end{aligned} \tag{E.6}$$

Comparing the results from the two coordinates, we conclude that the surface area Ω_D is

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \, . \tag{E.7}$$

The expression for Ω_D in terms of the Gamma function is obtained only for integer values of D, but we can use this expression for the analytic continuation to complex D dimensions. Therefore, when the integrand of a D dimensional integral is only a function of p^2, we have

$$\int_{-\infty}^{\infty} d^D p\, f(p^2) = \int d\Omega_D \int_0^{+\infty} dp\, p^{D-1} f(p^2) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^{+\infty} dp\, p^{D-1} f(p^2)\,. \tag{E.8}$$

The gamma function is the extension of the factorial function to real and complex numbers such that $\Gamma(n) = (n-1)!$ for any positive integer n. The gamma function is defined for all complex numbers except the non-positive integers. For complex numbers with a positive real part, it is defined by the convergent integral

$$\Gamma(z) = \int_0^{\infty} dt\; t^{z-1}\; e^{-t}\;. \tag{E.9}$$

Using integration by part, one can show that $\Gamma(z) = (z-1)\Gamma(z-1)$. Combining this with $\Gamma(1) = 1$, one can see that $\Gamma(n) = (n-1)!$ for all positive integer n. The property $\Gamma(z) = (z-1)\Gamma(z-1)$ can be used to calculate $\Gamma(z)$, where z is a complex number with a positive real part, in terms of $\Gamma(z')$, where $1 \leq \Re(z') < 2$ and $z' = z + n$ with an integer n.

The integral in eq (E.9) is convergent only for complex numbers with positive real parts. However, the gamma function can be extended to the complex numbers with non-positive real parts as well. This can be done by analytic continuation. Then one can see that the relation $\Gamma(z-1) = \Gamma(z)/(z-1)$ holds for all complex numbers. Therefore $\Gamma(z)$ is an analytic function of z that has simple poles at non-positive integers.

Here are some useful relations:

$$\begin{aligned}
\Gamma(\frac{1}{2}) &= \sqrt{\pi} \\
\Gamma(1)) &= 1 \\
\Gamma(\frac{3}{2}) &= \frac{1}{2}\sqrt{\pi} \\
\Gamma(z) &= \frac{\Gamma(z+1)}{z} \\
\text{Res.}\,[\Gamma(z), z=-n] &= \frac{(-1)^n}{n!} \\
\Gamma(\epsilon) &= \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \\
\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma) + \mathcal{O}(\epsilon)\right]\,.
\end{aligned} \tag{E.10}$$