T30f
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Quantum Field Theory

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Contents

1 Path Integral Quantization

1.1 Non-relativistic Quantum Mechanics

Let us start with a conceptual discussion, namely the *double slit* experiment (see Fig. 1.1). A particle emitted from a source S at time $t = 0$ passes through one or the other of the two holes A_1 and A_2 drilled in a screen, and it is detected at time $t = T$ by a detector located at O.

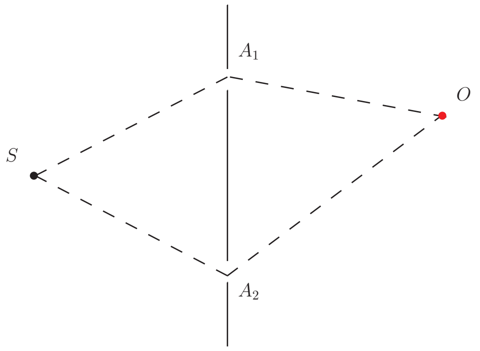


Figure 1.1: Double slit experiment.

According to the postulates of Quantum Mechanics (QM) we may say:

- The probability for the detection in O is given by

  $$P(S \to O) = |A(S \to O)|^2 \,, \tag{1.1}$$

  where A is the amplitude of the process.

- From the superposition principle the amplitude for the detection in O is the sum of the amplitude for the particle to propagate from the source S through the hole A_1 and then onward to the point O, and the amplitude for the particle to propagate from S through A_2 and then to O:

  $$A(S \to O) = A(S \to A_1 \to O) + A(S \to A_2 \to O) \,. \tag{1.2}$$

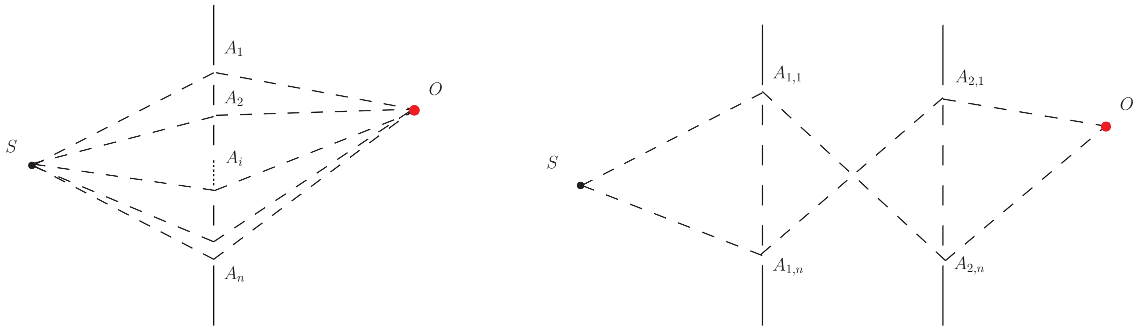


Figure 1.2: Experiment with one screen with n holes (left), experiment with two screens with n holes each (right).

We note that the corresponding probability contains the interference between the two terms in Eq. (1.2). If we drill n holes (see Fig. 1.2) we find for the amplitude the generalization of Eq. (1.2), that reads

$$A(S \to O) = \sum_{i=1}^{n} A(S \to A_i \to O)\,. \tag{1.3}$$

If we add a second screen with n holes (see again Fig. 1.2) we have for the amplitude

$$A(S \to O) = \sum_{i,j=1}^{n} A(S \to A_{1,i} \to A_{2,j} \to O)\,. \tag{1.4}$$

If one would add an infinite numbers of screens, drill infinite holes in each, we will get

$$A(S \to O) = \sum_{\text{all paths}} A(S \to \text{path} \to O)\,, \tag{1.5}$$

that is understood as a sum over all paths (see Fig. 1.3). Hence in order to calculate the amplitude we have to sum over the amplitude for the particle to propagate from the source to the detector following *all* possible paths.

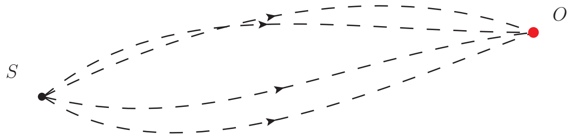
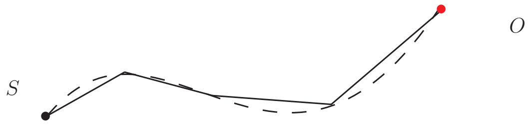


Figure 1.3: All possible paths in the limit of infinite holes and screens.

1) *How could one define the sum over the paths?*
   We have to consider the functional integral that is a sum over an infinite number of possible trajectories in order to compute a quantum amplitude. We can take a path and approximate it by straight line segments and let the segments go to zero.



2) *How can we construct the amplitude along a particular path?*
   If we know the amplitude for each infinitesimal segment, then we can just multiply them together to get the amplitude of the whole path.

In QM the amplitude to propagate from a point q_I to a point q_F in a time T is governed by the unitary operator $e^{-i\hat{H}T}$, where $\hat{H}$ is the Hamiltonian. The corresponding amplitude is given by $\langle q_F|e^{-i\hat{H}T}|q_I\rangle$.

1.2 Transition amplitude as path integrals

Feynman, following the ideas from Dirac, has shown that QM could be formulated in terms of path integrals. We discuss this approach to QM in details since it provides the key to the path integral formulation of QFT.

We consider a quantum mechanical system with one degree of freedom: one generalized coordinate, x, and its conjugate momentum, p. We keep $\hbar \neq 1$ and $c = 1$. In the canonical quantization we work with the Hilbert space operators $\hat{x}$ and $\hat{p}$ defined by the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar\,, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})\,. \tag{1.6}$$

where we also display the Hamiltonian of the system under consideration. The eigenstates of the position operator are introduced as:

$$\begin{cases} \hat{x}_H(t)|x,t\rangle_H = x|x,t\rangle_H\,, & \text{Heisenberg picture} \\ \hat{x}_S|x\rangle_S = x|x\rangle_S\,, & \text{Schrödinger picture} \end{cases} \tag{1.7}$$

with the relation between the two pictures that reads

$$\begin{cases} |x,t\rangle_H = e^{\frac{i\hat{H}t}{\hbar}}|x\rangle_S \\ |x\rangle_S = e^{-\frac{i\hat{H}t}{\hbar}}|x,t\rangle_H. \end{cases} \tag{1.8}$$

Since $\hat{x}_H(t)$ is time dependent, so are the eigenstates

$$\hat{x}_H(t) = e^{\frac{i\hat{H}t}{\hbar}}\,\hat{x}_S\,e^{-\frac{i\hat{H}t}{\hbar}}\,. \tag{1.9}$$

The eigenstates are normalized as follows:

$$\begin{cases} \langle x''|x'\rangle = \delta(x''-x')\,, & \int_{-\infty}^{+\infty} dx'|x'\rangle\langle x'| = 1\,, \\ \langle p''|p'\rangle = \delta(p''-p')\,, & \int_{-\infty}^{+\infty} dp'|p'\rangle\langle p'| = 1\,, \\ \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p'x'}{\hbar}}\,. \end{cases} \tag{1.10}$$

The matrix element ${}_H\langle x',t'|x,t\rangle_H = {}_S\langle x'|e^{-i\frac{\hat{H}(t'-t)}{\hbar}}|x\rangle_S$ corresponds to the transition from the eigenstate $|x,t\rangle$ at time t to the eigenstate $|x',t'\rangle$ at time t', hence it gives the *transition amplitude.*

This matrix element can be first expressed as a *multiple integral* which will then be used to define the *functional integral* (path integral) via a limiting procedure.

First we divide the time interval $(t'-t)$, characterizing the two states, into $(n+1)$ parts of equal lengths, ϵ, as follows

$$\begin{cases} (n+1)\epsilon = (t'-t) \\ t' = (n+1)\epsilon + t \\ t_j = j\epsilon + t\,, \quad j = 1, ..., n\,. \end{cases} \tag{1.11}$$

and $x_0 = x$ for $t_0 = t$, and $x_{n+1} = x'$ for $t_{n+1} = t'$ are kept fixed. Then we can use the completeness relation at each of the time t_j:

$$\int dx_j\,|x_j,t_j\rangle_{HH}\langle x_j,t_j| = 1\,, \tag{1.12}$$

to derive

$$
\begin{aligned}
{}_H\langle x', t'|x, t\rangle_H &= \int dx_1 \cdots dx_n \; {}_H\langle x', t'|x_n, t_n\rangle_H \; {}_H\langle x_n, t_n|x_{n-1}\,, t_{n-1}\rangle_H \\
&\qquad \cdots {}_H\langle x_{j+1},\, t_{j+1}|x_j,\, t_j\rangle_H \cdots {}_H\langle x_1, t_1|x\,, t\rangle_H
\end{aligned} \tag{1.13}
$$

Now we proceed using the following relation

$$
{}_H\langle x_j, t_j|x_{j-1}, t_{j-1}\rangle_H = \langle x_j|e^{-i\frac{\epsilon \hat{H}}{\hbar}}|x_{j-1}\rangle = \langle x_j|x_{j-1}\rangle - i\frac{\epsilon}{\hbar}\langle x_j|\hat{H}|x_{j-1}\rangle + \mathcal{O}(\epsilon^2)\;, \tag{1.14}
$$

where

$$
\begin{cases} x_0 = x,\; t_0 = t \\ x_{n+1} = x',\; t_{n+1} = t'. \end{cases} \tag{1.15}
$$

are fixed. We choose a general form for the Hamiltonian (separable)

$$
\hat{H}(x,p) = \hat{f}(p) + \hat{g}(x), \tag{1.16}
$$

so that we can write

$$
\langle x_j|\hat{H}|x_{j-1}\rangle = \int dp_j \langle x_j|p_j\rangle\langle p_j|\hat{H}|x_{j-1}\rangle = \int \frac{dp_j}{2\pi\hbar} exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1})\right] H(p_j, x_{j-1})\,, \tag{1.17}
$$

and now $H(p_j, x_{j-1})$ is a classical number and not an operator any more. Inserting Eq. (1.17) inside Eq. (1.14) we obtain

$$
\begin{aligned}
{}_H\langle x_j, t_j|x_{j-1}, t_{j-1}\rangle_H &= \int \frac{dp_j}{2\pi\hbar}\, exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1})\right]\left[1 - \frac{i}{\hbar}\epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2) \\
&= \int \frac{dp_j}{2\pi\hbar}\, exp\left[\frac{i}{\hbar}p_j(x_j - x_{j-1}) - \frac{i}{\hbar}\epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2)\,,
\end{aligned} \tag{1.18}
$$

and composing the infinitesimal transition amplitudes in (1.13) using eqs. (1.14) and (1.18) we find

$$
{}_H\langle x', t'|x, t\rangle_H = \lim_{n\to\infty} \int \prod_{j=1}^{n} dx_j \int \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} exp\left\{\frac{i}{\hbar}\sum_{j=1}^{n+1}\left[p_j(x_j - x_{j-1}) - H(p_j, x_{j-1})(t_j - t_{j-1})\right]\right\} \tag{1.19}
$$

where the limit $n \to \infty$ ($\epsilon \to 0$) has been taken and terms of order $\mathcal{O}(\epsilon^2)$ neglected. So we obtain the transition amplitude as a path integral

$$
{}_H\langle x', t'|x, t\rangle_H = \int \mathcal{D}x\mathcal{D}p\, exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\, \left[p\dot{x} - H(p,x)\right]\right\}\,, \tag{1.20}
$$

with $x(t) = x$ and $x(t') = x'$ fixed. The right-hand side in (1.20) is called functional integral over the phase space

$$
\mathcal{D}x = \prod_{j=1}^{n} dx_j\,, \quad \mathcal{D}p = \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} \tag{1.21}
$$

in the limit $n \to \infty$.

If the Hamiltonian is of the simple form

$$
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})\,, \tag{1.22}
$$

it is convenient to perform the p integration in (1.18). We define $\Delta x_j = x_j - x_{j-1}$ and we obtain

$$\int \frac{dp_j}{2\pi\hbar} exp\left\{\frac{i}{\hbar}\left(p_j\Delta x_j - \frac{p_j^2}{2m}\epsilon\right)\right\} = \frac{1}{2\pi\hbar}\sqrt{\frac{\pi 2m\hbar}{i\epsilon}} exp\left\{-\frac{(\Delta x_j)^2\hbar 2m}{4\hbar^2\epsilon i}\right\}$$
$$= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i\epsilon}} exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\Delta x_j}{\epsilon}\right)^2\epsilon\right\} = \frac{1}{C_j} exp\left\{\frac{i}{\hbar}\frac{m}{2}\left(\frac{\Delta x_j}{\epsilon}\right)^2\epsilon\right\}. \quad (1.23)$$

We used the result for the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2+bx} = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{b^2}{4\alpha}}. \quad (1.24)$$

Such result can be equivalently obtained making the shift $p_j \to p_j + \frac{m\Delta x_j}{\epsilon}$, and the prefactor of the exponential in (1.23) reads

$$\frac{1}{C_j} = \int \frac{dp_j}{2\pi\hbar} exp\left\{-\frac{i}{\hbar}\frac{p_j^2}{2m}\epsilon\right\} = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i\epsilon}} \quad (1.25)$$

which is divergent in the limit $\epsilon \to 0$. However it is compensated by an analogous factor in the path integral. In this way the final result has the form of a functional integral over configuration space:

$$_H\langle x', t'|x, t\rangle_H = \lim_{n\to\infty}\frac{1}{C}\int\prod_{j=1}^{n} dx_j \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x_j - x_{j-1})^2}{(t_j - t_{j-1})^2}(t_j - t_{j-1}) - V(x_{j-1})(t_j - t_{j-1})\right]\right\}, \quad (1.26)$$

which reads

$$_H\langle x', t'|x, t\rangle_H = \frac{1}{C}\int \mathcal{D}x\, exp\left\{\frac{i}{\hbar}S[x]\right\}, \quad (1.27)$$

where $S[x] = \int_t^{t'} L(x, \dot{x})d\tau$ is the action integral over the trajectory $x(\tau)$ with the Lagrangian $L(x, \dot{x}) = m\dot{x}^2/2 - V(x)$. The normalization factor is given by

$$\frac{1}{C} = \prod_{j=1}^{N}\frac{1}{C_j} = \int \mathcal{D}p\, exp\left\{-\frac{i}{\hbar}\int_t^{t'}\frac{p^2}{2m}d\tau\right\}. \quad (1.28)$$

So we finally derived (1.27) *starting from a canonically quantized theory described by the Hamiltonian* (1.22). We can use another approach, define the quantum theory by the functional integral in (1.27), or in other words we can choose the path integral formulation as the quantization prescription for a system with the classical Hamiltonian in the form of (1.22). Then our derivation proves the equivalence of the *path integral* and *canonical quantization* methods for systems described by the Hamiltonian in (1.22).

From now on we will consider quantum theories defined by the path integral formulation. Let us consider

$$_H\langle x', t'|x, t\rangle_H = \frac{1}{C}\int \mathcal{D}x\, exp\left\{\frac{i}{\hbar}S[x]\right\}, \quad (1.29)$$

if happens that

$$S[x(\tau)] \gg \hbar, \text{ i.e.}\, \hbar \to 0 \quad (1.30)$$

we can evaluate (1.29) by using the saddle point approximation. For $\hbar \to 0$ all the configurations far from the extrema of S give contributions that oscillate wildly under small deformations of the path and thus give zero contribution. The only first non-vanishing contribution comes from the extrema of the action,

$$\frac{\delta S}{\delta x} = 0\,, \tag{1.31}$$

which leads to the Euler–Lagrange equation

$$\frac{\partial L(x,\dot{x})}{\partial x} - \frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}} = 0\,. \tag{1.32}$$

This is the equation that defines the classical trajectory and the classical equations of motion in classical mechanics.

To obtain the semi-classical solution, one can use the power series expansion of the functional $S[x]$ about its classical solution

$$\begin{aligned} S[x] &= S[x_{\rm cl}] + S'[x](x-x_{\rm cl}) + \frac{1}{2}S''[x](x-x_{\rm cl})^2 + \mathcal{O}(x-x_{\rm cl})^3 \\ &= S[x_{\rm cl}] + \frac{1}{2}S''[x](x-x_{\rm cl})^2 + \mathcal{O}(x-x_{\rm cl})^3\,, \end{aligned} \tag{1.33}$$

where

$$\begin{aligned} S'[x] &= \frac{\delta S}{\delta x} \\ S''[x] &= \frac{\delta^2 S}{\delta x^2}\,. \end{aligned} \tag{1.34}$$

Truncating the above expansion and putting it in Eq. 1.29, one can see that

$$\frac{1}{C}\int \mathcal{D}x\, \exp\left\{\frac{i}{\hbar}S\left[x\right]\right\} \to \frac{1}{C}\exp\left\{\frac{i}{\hbar}S\left[x_{\rm cl}\right]\right\}\int \mathcal{D}x\, \exp\left\{\frac{i}{2\hbar}S''[x](x-x_{\rm cl})^2\right\}\,. \tag{1.35}$$

The term $\int \mathcal{D}x\, \exp\left\{\frac{i}{2\hbar}S''[x](x-x_{\rm cl})^2\right\}$ is the semi-classical contribution to the path integral. As a matter of fact, the path integral formulation allows a relatively simple understanding of the classical and semi-classical limit. As we increase $\hbar$ the classical trajectory still dominates but there are other paths close to it whose action is within $\Delta S \simeq \hbar$ and contribute significantly to the amplitude. The particle does an excursion around the classical trajectory.

The matrix element ${}_H\langle x',t'|x,t\rangle_H$ determines all the transition probabilities between quantum mechanical states. In view of the applications of the functional formalism to quantum field theories, it is important to know the path integral representation of the *matrix element of the position operators* that corresponds to the field operators of QFT.

For the time-ordered product of n such operators holds the formula

$${}_H\langle x',t'|T\hat{x}(t_1)\ldots\hat{x}(t_n)|x,t\rangle_H = \int \mathcal{D}x\mathcal{D}p\; x(t_1)\ldots x(t_n)\, e^{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{x}-H(p,x)]}\,. \tag{1.36}$$

Here the time-ordered product is defined as

$$T\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n) = \hat{x}(\tau_1)\hat{x}(\tau_2)\ldots\hat{x}(\tau_n), \tag{1.37}$$

where $\tau_i > \tau_{i+1}$ and $\tau_i = t_j$. We can show that (1.36) is true for the case of two operators $\hat{x}(\tau_1)\hat{x}(\tau_2)$ for $\tau_1 > \tau_2$. We divide again the time interval $(t' - t)$ into small intervals choosing $t_1 \dots t_n$ such that

$$\tau_1 = t_{i_1}\,, \quad \tau_2 = t_{i_2}\,, \tag{1.38}$$

and apply the completeness relation at each t_i. We have

$$\begin{aligned} {}_H\langle x', t'|\hat{x}(\tau_1)\hat{x}(\tau_2)|x, t\rangle_H &= \int \prod_i dx_i\ {}_H\langle x', t'|x_n, t_n\rangle_H \cdots {}_H\langle x_{i_1}, t_{i_1}|\hat{x}(\tau_1)|x_{i_1-1}\,, t_{i_1-1}\rangle_H \\ &\qquad \cdots {}_H\langle x_{i_2},\, t_{i_2}|\hat{x}(\tau_2)|x_{i_2-1},\, t_{i_2-1}\rangle_H \cdots {}_H\langle x_1, t_1|x\,, t\rangle_H \\ &= \int \prod_i dx_i x_{i_1} x_{i_2} {}_H\langle x', t'|x_n, t_n\rangle_H \cdots {}_H\langle x_1, t_1|x, t\rangle_H\,. \end{aligned} \tag{1.39}$$

Then if we proceed exactly as before we end up with (1.36). In particular, Eq. (1.39) is true when $\tau_1 > \tau_2$. When $\tau_2 > \tau_1$ the left-hand side of (1.39) is ${}_H\langle x', t'|\hat{x}(\tau_2)\hat{x}(\tau_1)|x, t\rangle_H$. So the path integral is equal to

$$\int \mathcal{D}x\mathcal{D}p\ x(\tau_1)x(\tau_2)\, e^{\frac{i}{\hbar}\int_t^{t'} d\tau[p\dot{x} - H(p,x)]} = \begin{cases} {}_H\langle x', t'|\hat{x}(\tau_1)\hat{x}(\tau_2)|x, t\rangle_H\,,\ \tau_1 > \tau_2 \\ {}_H\langle x', t'|\hat{x}(\tau_2)\hat{x}(\tau_1)|x, t\rangle_H\,,\ \tau_2 > \tau_1\,. \end{cases} \tag{1.40}$$

As before it is possible to go from phase space path integrals to the path integrals over configuration space.

1.2.1 Transition amplitude in the presence of an external source $J(\tau)$

The transition amplitude with an external source reads

$${}_H\langle x', t'|x, t\rangle_H^J = \int \mathcal{D}x\mathcal{D}p\, exp\left\{\frac{i}{\hbar}\int_t^{t'} d\tau\, [p\dot{x} - H(p,x) + \hbar J(\tau)x(\tau)]\right\}\,, \tag{1.41}$$

where we have modified the Hamiltonian with a source term $H \to H - \hbar Jx$. This transition amplitude in presence of an external source can be used as *generating functional* of the matrix element of the position operators. They are given by its functional derivatives with respect to $J(\tau)$:

$${}_H\langle x', t'|T\hat{x}(t_1)\dots\hat{x}(t_n)|x, t\rangle_H = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\, {}_H\langle x', t'|x, t\rangle_H^J \Bigg|_{J=0} \tag{1.42}$$

where the $\delta/\delta J(t)$ are the functional derivatives.

1.3 Vacuum to vacuum transitions in imaginary time formalism

In this section we do not explicitly indicate the Heisenberg vectors with the subscript H in order not to make the formulas too cumbersome. However all the bras and kets where both the position and the time appear are understood as bras and kets in the Heisenberg picture. In the case when the time appears in the bra and the ket, those are in the Heisenberg representation even if it is not declared with superscripts.

In QFT we are interested in the calculation of Green's functions that are matrix elements of field operators taken between *vacuum states*. It is then useful to consider the analogous problem in QM.
We assume that the Lagrangian of the system is time independent. The energy eigenstates are defined as $\hat{H}|n\rangle = E_n|n\rangle$. In configuration space (Schrödinger description) they correspond to the wave functions $\phi_n(x) = \langle x|n\rangle$. The ground state or vacuum state is described by the wave function $\phi_0(x) = \langle x|0\rangle$.
We have

$$\phi_0(t,x) = e^{-\frac{i}{\hbar}E_0 t}{}_S\langle x|0\rangle = \langle x|e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle = {}_H\langle x,t|0\rangle \,. \tag{1.43}$$

We are interested in $\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle$. It can be written, by inserting two completeness relations $\int dx'\,|x',t'\rangle\langle x',t'| = 1$ and $\int dx\,|x,t\rangle\langle x,t| = 1$, as follows

$$\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle = \int dx'\,dx\,\phi_0^*(t',x')\langle x',t'|T\hat{x}(t_1)\ldots\hat{x}(t_n)|x,t\rangle\phi_0(t,x) \tag{1.44}$$

and for the transition amplitude in (1.44) we can use the result given in (1.36). The vacuum expectation value written in (1.44) can also be obtained from a generating functional

$$\langle 0|T\hat{x}(t_1)\ldots\hat{x}(t_n)|0\rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)} Z[J]\Bigg|_{J=0} \tag{1.45}$$

with

$$Z[J] = \langle 0|0\rangle^J = \int dx'\,dx\,\phi_0^*(t',x')\langle x',t'|x,t\rangle^J\phi_0(t,x)\,. \tag{1.46}$$

One may understand the definition of the generating functional without the external current as follows

$$\langle 0|\int dx'\,|x',t'\rangle\langle x',t'|\int dx\,|x,t\rangle\langle x,t|0\rangle = \int dx'dx\phi_0^*(t',x')\phi_0(t,x)\langle x',t'|x,t\rangle = Z[0]\,, \tag{1.47}$$

then one can just replace $\langle x',t'|x,t\rangle$ with $\langle x',t'|x,t\rangle^J$ that is in turn given in (1.41).

It is very important to derive the generating functional $Z[J]$ in a different way. We are going to show that

$$Z[J] = \lim_{\substack{T_1\to+i\infty\\ T_2\to-i\infty}} \frac{exp\left[(i/\hbar)E_0(T_2-T_1)\right]}{\phi_0^*(x_1)\phi_0(x_2)}\langle x_2,T_2|x_1,T_1\rangle^J\,. \tag{1.48}$$

This implies that $Z[J]$ for $T_1 \to +i\infty$ and $T_2 \to -i\infty$ is determined by the transition amplitude $\langle x_2,T_2|x_1,T_1\rangle^J$ *at any given* x_1,x_2 (for example $x_1 = x_2 = 0$, it does not matter which values we choose for x_1 and x_2) provided that the analytic continuation to the imaginary values of T_1 and T_2 is done.

To prove eq. (1.48) we choose a source $J(\tau)$ that vanishes outside the time interval (t,t') with $T_2 > t' > t > T_1$. Then we can write

$$\langle x_2,T_2|x_1,T_1\rangle^J = \int dx'dx\langle x_2,T_2|x',t'\rangle\langle x',t'|x,t\rangle^J\langle x,t|x_1,T_1\rangle \tag{1.49}$$

where all the states are understood in the Heisenberg picture and we write

$${}_H\langle x,t|x_1,T_1\rangle_H = \langle x|e^{-\frac{i}{\hbar}H(t-T_1)}|x_1\rangle =$$

$$
\begin{aligned}
&= \sum_n \langle x | e^{-\frac{i}{\hbar}H(t-T_1)} | n\rangle \langle n | x_1\rangle \\
&= \sum_n \sum_{n'} \langle x | n'\rangle \langle n' | e^{-\frac{i}{\hbar}H(t-T_1)} | n\rangle \langle n | x_1\rangle = \\
&= \sum_n \langle x | n\rangle \langle n | x_1\rangle e^{-\frac{i}{\hbar}E_n(t-T_1)} \\
&= \sum_n \phi_n(x)\phi_n^*(x_1) e^{-\frac{i}{\hbar}E_n(t-T_1)} \longrightarrow \phi_0(x)\phi_0^*(x_1) e^{-\frac{i}{\hbar}E_0(t-T_1)} \qquad (1.50)
\end{aligned}
$$

where we have inserted two complete set of eigenstates to obtain the result. Similarly we can do the same for ${}_H\langle x_2, T_2 | x', t'\rangle_H$. The only T-dependent terms is now the factor $e^{-\frac{i}{\hbar}E_n(t-T_1)}$ and we can continue $T \to +i\infty$ explicitly.

The oscillatory behaviour of the exponential is not well defined and we need to make some more precise statement in order to evaluate it unambiguously. To define it one can continue the time to imaginary values and then continue back to the real time at the end ($t = i\tau$).

In the limit $T \to \infty$ in the sum $\sum_n e^{-E_n T}$ only E_0 which is the lowest energy level survives. For this reason we can write, by using the result in (1.50) :

$$
\lim_{T_1 \to +i\infty} e^{-\frac{i}{\hbar}E_0 T_1} \langle x, t | x_1, T_1\rangle = \phi_0(x) e^{-\frac{i}{\hbar}E_0 t} \phi_0^*(x_1) = \phi_0(t, x)\phi_0^*(x_1) \, . \qquad (1.51)
$$

In the same way we calculate

$$
\lim_{T_2 \to -i\infty} e^{\frac{i}{\hbar}E_0 T_2} \langle x_2, T_2 | x', t'\rangle = \phi_0^*(t', x')\phi_0(x_2) \, . \qquad (1.52)
$$

Then using (1.51) and (1.52) and the definition (1.46) in (1.49) we obtain

$$
\begin{aligned}
&\langle x_2, T_2 | x_1, T_1\rangle^J = \int dx' dx \langle x_2, T_2 | x', t'\rangle \langle x', t' | x, t\rangle^J \langle x, t | x_1, T_1\rangle \\
&= \int dx' dx e^{-\frac{i}{\hbar}E_0 T_2} \phi_0^*(t', x')\phi_0(x_2) \langle x', t' | x, t\rangle^J e^{\frac{i}{\hbar}E_0 T_1} \phi_0(x, t)\phi^*(x_1) \\
&= e^{-\frac{i}{\hbar}E_0 (T_2 - T_1)} \phi_0(x_2)\phi^*(x_1) Z[J] \, . \qquad (1.53)
\end{aligned}
$$

We have therefore obtained (1.48) back and we have proven that $Z[J]$ can be given also in terms of (1.48).

Then if (1.48) is true we will also have

$$
\langle 0 | T x(t_1) \ldots x(t_n) | 0\rangle = \lim_{\substack{T_1 \to +i\infty \\ T_2 \to -i\infty}} \frac{e^{\frac{i}{\hbar}E_0 (T_2 - T_1)}}{\phi_0^*(x_1)\phi_0(x_2)} \langle x_2, T_2 | T x(t_1) \cdots x(t_n) | x_1, T_1\rangle \, . \qquad (1.54)
$$

Then the vacuum matrix elements can be calculated by taking functional derivatives of the generating functional $Z[J]$ given by (1.48). The J-independent factors are irrelevant because we can always consider quantities like

$$
\frac{1}{\langle 0 | 0\rangle} \langle 0 | T x(t_1) \ldots x(t_n) | 0\rangle = \left(\frac{1}{i}\right)^n \frac{1}{Z[0]} \frac{\delta^n}{\delta J(t_1) \cdots \delta J(t_n)} Z[J] \bigg|_{J=0} \, . \qquad (1.55)
$$

Then we can simply write in place of (1.48):

$$
Z[J] = const \lim_{\substack{T_1 \to +i\infty \\ T_2 \to -i\infty}} \int \mathcal{D}x \, exp \left\{ \frac{i}{\hbar} \int_{T_1}^{T_2} [L(x, \dot{x}) + \hbar J x] \, dt \right\} \, , \qquad (1.56)
$$

where x_1 and x_2 are arbitrary. The path integral is over all $x(t)$ satisfying the boundary conditions $\lim_{T_1\to i\infty} x(T_1) = x_1$ and $\lim_{T_2\to -i\infty} x(T_2) = x_2$. Moreover x_1 and x_2 are any chosen constants but often are taken to be zero.

1.4 Path integral formulation of Quantum Field Theory

Green's functions as path integrals

The results that we have obtained up to now are simple to generalize to the case of more than one degree of freedom. If *the number of degrees of freedom is to be d*, the coordinate x should be replaced by a d-component vector. The functional integral now corresponds to the sum over all trajectories in the d-dimensional configuration space, satisfying appropriate boundary conditions. For example, for $p \to \mathbf{p}$ and $x \to \mathbf{x}$ in $d = 3$ we have

$$\prod_{k=1}^{N+1} dp_k \to \prod_{k=1}^{N+1} dp_k^1 \prod_{k=1}^{N+1} dp_k^2 \prod_{k=1}^{N+1} dp_k^3 \,, \tag{1.57}$$

$$\prod_{h=1}^{N} dx_h \to \prod_{h=1}^{N} dx_h^1 \prod_{h=1}^{N} dx_h^2 \prod_{h=1}^{N} dx_h^3 \,. \tag{1.58}$$

In Field Theory, the trajectory $x(t)$ is replaced by a field function $\phi(t, \mathbf{x})$. The degrees of freedom are labelled by a continuous index $\mathbf{x}$, the number of degrees of freedom is *infinite.* In this case to define the appropriate path integral one can start from a *multiple* integral on a *discrete*, and for the moment *finite*, lattice of space-time points.

Here we need to insert the definition of the discretized generating functional. To this aim one divides the space into 4-dimensional cubes of volume ε^4 and identify each degree of freedom with discrete labels, that in turn enters the field and its derivative as follows

$$\phi_n \simeq \phi(t_l, x_i, y_j, z_k) \,, \tag{1.59}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{l,i,j,k} \simeq \frac{1}{\varepsilon}\left[\phi(t_l, x_i+\varepsilon, y_j, z_k) - \phi(t_l, x_i, y_j, z_k)\right] \,. \tag{1.60}$$

If then we put $n = l, i, j, k$ we can write

$$\mathcal{L}(\phi(t_l, x_i, y_j, z_k), \partial_\mu\phi(t_l, x_i, y_j, z_k)) = \mathcal{L}(\phi_n, \partial_\mu\phi_n) = \mathcal{L}_n \,, \tag{1.61}$$

and also for the generating functional

$$Z[J] = \lim_{N\to\infty}\lim_{\varepsilon\to 0} \int \prod_{n}^{N^4} d\phi_n exp\left\{ i \sum_{n=1}^{N^4} \varepsilon^4 (\mathcal{L}_n + \hbar J_n \phi_n) \right\} \,. \tag{1.62}$$

This corresponds to defining QFT as a limit of a theory with only a finite number of degrees of freedom. The limit of an infinite lattice, which is related to the thermodynamical limit of statistical mechanics, already defines a theory with an infinite number of degrees of freedom.

However, this lattice theory does not have the usual Lorentz space-time invariance (it has a different symmetry in space-time) and one has to send the lattice step to zero and go to the continuous limit. This continuous limit is accompanied by infinities, the *ultraviolet* (UV) divergences of QFT (UV divergences → related to high energies/small distances). Then the definition of the functional integral in QFT is more ambiguous than in the case of quantum mechanics.

QFT is formulated in terms of vacuum expectation values of time-ordered products of field operators, the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \langle 0|T\phi(x_1)...\phi(x_n)|0\rangle \,. \tag{1.63}$$

We will write down the path integral representation for the $G^{(n)}$ using what we have developed up to here in QM. It is particularly important to remember the role played by Eqs. (1.48) and (1.54) in getting rid of the vacuum wave functions that are originally present in (1.44) as boundary conditions.

By analogy with the previous result we can postulate this path integral representation:

$$G^{(n)}(x_1, \dots, x_n) \sim \int \mathcal{D}\phi \, \phi(x_1)...\phi(x_n) e^{\frac{i}{\hbar}\int d^4x\mathcal{L}} \,, \tag{1.64}$$

where $\mathcal{D}\phi$ denotes the integration over all the functions $\phi(t, \mathbf{x})$ of space and time because for each value of $\mathbf{x}$, $\phi(t, \mathbf{x})$ corresponds to a separate degree of freedom. Moreover, the path integral is understood over all functions that satisfy the boundary conditions

$$\begin{cases} \phi(T_1, \mathbf{x}) = \phi_1(x) \,, \\ \phi(T_2, \mathbf{x}) = \phi_2(x) \,, \end{cases} \tag{1.65}$$

for $T_1 \to +i\infty$ and $T_2 \to -i\infty$, and $\phi_1(x)$ and $\phi_2(x)$ are arbitrary and can be put to zero. In fact (1.54) suggests that the boundary conditions are irrelevant.

In Eq. (1.64) we have an oscillatory behaviour. To make well defined calculations we can proceed in two ways:

1) Wick rotation: analytic continuation to imaginary time region, namely

$$\tau = -it \,, \quad t = i\tau \,. \tag{1.66}$$

In this way one obtains a particularly convenient path integral representation because the weight factor in the integral, $exp\,(-S_E/\hbar)$, is non-negative. Indeed we have that

$$S_E[\phi] = \int_E d^4x_E \mathcal{L}(x_E) = -i(-1)\int_M d^4x(-\mathcal{L}(x)) = -iS[\phi] \tag{1.67}$$

after we notice that $d^4x_E = -id^4x$ and

$$\mathcal{L}(x_E) = \frac{1}{2}\left[(\partial^E_\mu\phi)^2 + m^2\phi^2\right] = -\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) = -\mathcal{L} \,. \tag{1.68}$$

Then the Euclidean path integral formalism can be used to define Minkowski space Green's functions by analytic continuation of the Euclidean ones. So (1.64) may be understood as an analytic continuation in the variables $t_1, ..., t_n$ of the analogous Euclidean formula.

2) One can work in Minkowski space and regularize the integrand in the path integral by adding a small imaginary piece to the Lagrangian density $\mathcal{L}$ ($+i\epsilon$ prescription)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \frac{1}{2}i\epsilon\phi^2 \tag{1.69}$$

Eq. (1.64) has to be regarded as the formulation of the theory.

Now, what is the relation between this path integral definition and the usual canonical operator formulation of the QFT based on the same $\mathcal{L}$?

For our present scopes: they are equivalent if they originate the same perturbation theory and thus the same Feynman rules. We will check this in the following. The derivation of perturbation theory is much simpler in the functional framework, especially in the case of gauge field theories.

It is convenient to normalize the Green's functions by factorizing out the vacuum amplitude:

$$G^{(n)}(x_1, \dots, x_n) = \frac{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle}{\langle 0|0\rangle} = N\int \mathcal{D}\phi\,\phi(x_1)\dots\phi(x_n)exp\left\{\frac{i}{\hbar}\int d^4x\mathcal{L}\right\}, \tag{1.70}$$

where

$$\frac{1}{N} = \int \mathcal{D}\phi exp\left\{\frac{i}{\hbar}\int d^4x\mathcal{L}\right\} = \langle 0|0\rangle\,, \tag{1.71}$$

so that extra factors like those appearing in (1.54) are eliminated.

The Green's function in (1.70) are given *by the functional derivatives of the generating functional $Z[J]$ that gives the vacuum transition in the presence of an external sources*:

$$Z[J] = N\int \mathcal{D}\phi\,exp\left\{\frac{i}{\hbar}\int d^4x\left[\mathcal{L} + \hbar J(x)\phi(x)\right]\right\}, \tag{1.72}$$

and expanding $Z[J]$ in powers of J around $J = 0$ we obtain

$$Z[J] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int dx_1\cdots dx_n\,G^{(n)}(x_1, \dots, x_n)J(x_1)\cdots J(x_n) \tag{1.73}$$

and from this

$$G^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z[J]\bigg|_{J=0}\,. \tag{1.74}$$

The Green's functions can also be considered as the analytic continuation of those obtained from the generating functional defined in Euclidean space with $x_0 = ix_4$ ($t = i\tau$) with x_4 real:

$$Z_E[J] = N\int \mathcal{D}\phi\,exp\left\{-\frac{1}{\hbar}S_E[\phi(x)] + \int d^4x_E J(x)\phi(x)\right\}, \tag{1.75}$$

where $d^4x_E = dx_1\,dx_2\,dx_3\,dx_4$. For a scalar field

$$S_E[\phi(x_E)] = (1/2)\int d^4x_E\left[((\partial\phi)/(\partial x_4))^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{1.76}$$

we have

$$G_E^{(n)}(x_1 \dots x_n) = \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z_E[J]\bigg|_{J=0}\,. \tag{1.77}$$

Using the path integral formalism one can obtain the equations of motion (see exercise sheet No. 1), for example in free Euclidean scalar field theory.

Action quadratic in the fields

When the action is quadratic in the fields ($\hbar = 1$)

$$S = \frac{1}{2}\int d^4x d^4y \phi(x) A(x,y) \phi(y) \,, \tag{1.78}$$

the generating functional $Z[J]$ *can be calculated in a closed form.* Let us consider the example of the free Klein-Gordon theory that has an action quadratic in the fields. The Lagrangian density for the Klein-Gordon case is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \,, \tag{1.79}$$

and

$$Z[J] = N \int \mathcal{D}\phi \, exp\left\{ iS[\phi] + i\int d^4x J(x)\phi(x)\right\} \,. \tag{1.80}$$

We write explicitly the exponent as follows

$$\int d^4x \left[\mathcal{L}(\phi) + J\phi\right] = \int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2 + i\epsilon)\phi + J\phi\right] \,, \tag{1.81}$$

where we have integrated by parts to obtain the right hand side of Eq. (1.81). This equation can be rewritten as

$$\int d^4x \left[\mathcal{L}(\phi) + J\phi\right] = \int d^4x \int d^4y \left[\frac{1}{2}\phi(x)(-\partial_x^2 - m^2 + i\epsilon)\delta(x-y)\phi(y) + \int d^4x J(x)\phi(x)\right] \,, \tag{1.82}$$

where the subscript x in ∂_x^2 indicates that the partial derivatives are taken with respect to x. Comparing Eqs. (1.78) and (1.82), A can be identified as

$$A(x,y) = (-\partial_x^2 - m^2 + i\epsilon)\delta(x-y) \,. \tag{1.83}$$

Note that the prescription $+i\epsilon$ is introduced as a factor of convergence for the functional integral. We can complete the square by introducing a shifted field

$$\phi'(x) = \phi(x) - i\int d^4y D_F(x-y) J(y) \,, \tag{1.84}$$

where $D_F(x-y) = G^{(2)}(x-y)$ for the free particle and we shall exploit the relation

$$(\partial^2 + m^2 - i\epsilon) D_F(x-y) = -i\delta^{(4)}(x-y) \,. \tag{1.85}$$

Then we obtain

$$\begin{aligned}
&\int d^4x \left[\mathcal{L}(\phi) + J\phi\right] = \int d^4x \frac{1}{2}\left[\phi'(-\partial^2 - m^2 + i\epsilon)\phi'\right] \\
&+i\int d^4x \int d^4y \, \frac{1}{2}\left[\phi'(-\partial_x^2 - m^2 + i\epsilon) D_F(x-y) J(y)\right] \\
&+i\int d^4x \int d^4y \, \frac{1}{2}\left[D_F(x-y) J(y) (-\partial_x^2 - m^2 + i\epsilon)\phi'(x)\right] \\
&+\frac{(i)^2}{2}\int d^4x \int d^4y \int d^4z D_F(x-y) J(y) (-\partial^2 - m^2 + i\epsilon) D_F(x-z) J(z)
\end{aligned}$$

$$
\begin{aligned}
&+\int d^4xJ(x)\phi'(x)+i\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\\
&=\int d^4x\frac{1}{2}\left[\phi'(-\partial^2-m^2+i\epsilon)\phi'\right]+\frac{i}{2}\int d^4x\int d^4y\phi'(x)i\delta^{(4)}(x-y)J(y)\\
&+\frac{i}{2}\int d^4x\int d^4yi\delta^{(4)}(x-y)J(y)\phi'(x)-\frac{1}{2}\int d^4x\int d^4y\int d^4zD_F(x-y)J(y)i\delta^{(4)}(x-z)J(z)\\
&+\int d^4xJ(x)\phi'(x)+i\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\\
&=\int d^4x\frac{1}{2}\left[\phi'(-\partial^2-m^2+i\epsilon)\phi'\right]-\int d^4xJ(x)\phi'(x)+\int d^4xJ(x)\phi'(x)\\
&-\frac{i}{2}\int d^4x\int d^4yJ(x)D_F(x-y)J(y)+i\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\\
&=\int d^4x\frac{1}{2}\left[\phi'(-\partial^2-m^2+i\epsilon)\phi'\right]+\frac{i}{2}\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\,. \qquad (1.86)
\end{aligned}
$$

Now we change variable from ϕ to ϕ' in the functional integral in (1.80). Because this is a shift the Jacobian of the transformation is 1, so we obtain

$$
\begin{aligned}
&Z[J]=N\int\mathcal{D}\phi' e^{i\int d^4\mathcal{L}(\phi')}\,\exp\left\{-\frac{1}{2}\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\right\}\\
&\Rightarrow Z[J]=\exp\left\{-\frac{1}{2}\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\right\}\,. \qquad (1.87)
\end{aligned}
$$

To derive the above relation we used Eq. (1.71) to remove the normalization factor N. The functional generator is calculated in a closed form.

If we keep track of $\hbar$, we end up with

$$
Z[J]=\exp\left\{-\frac{1}{2\hbar}\hbar^2\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\right\}\,. \qquad (1.88)
$$

In summary, *all the free Green's functions can be obtained via derivative of (1.87).*

We can obtain the result in Eq. (1.87) by using the results in the Appendix about Gaussian integrals. Indeed we can exploit the expressions in Eq. (D.27) to directly calculate the generator functional $Z[J]$ in the case of free action of the scalar field.

We start from

$$
Z[J]=N\int\mathcal{D}\phi\,\exp\left\{i\int d^4x\left[\frac{1}{2}\phi(-\partial^2-m^2+i\epsilon)\phi+J\phi\right]\right\}\,, \qquad (1.89)
$$

and we know that $(\partial_x^2+m^2)D(x-y)=\delta^4(x-y)$ with $D(x-y)=iD_F(x-y)$. Then we have $D^{-1}(x-y)=(\partial_x^2+m^2)\delta^{(4)}(x-y)$, which satisfies the definition

$$
\int d^4z\,D^{-1}(x,z)D(z,y)=\delta^{(4)}(x-y). \qquad (1.90)
$$

This can be easily verified as $\int d^4z\,(\partial_x^2+m^2)\delta^{(4)}(x-z)D(z,y)=(\partial_x^2+m^2)D(x-y)=\delta^{(4)}(x-y)$. The integral in Eq. (1.89) is of the form of Eq. (D.28), with

$$
A=-i(-\partial^2-m^2+i\epsilon)\,,\quad A^{-1}=\frac{i}{-\partial^2-m^2+i\epsilon}=-iD(x-y)=+D_F(x-y)\,. \qquad (1.91)
$$

So we can calculate

$$
Z[J]=N\frac{1}{\sqrt{\det A}}\,\exp\left\{\frac{1}{2}\int d^4x\int d^4yiJ(x)D_F(x-y)iJ(y)\right\}
$$

$$= N\frac{1}{\sqrt{\det A}}\exp\left\{-\frac{1}{2}\int d^4x\int d^4y J(x)D_F(x-y)J(y)\right\}\,. \tag{1.92}$$

Moreover, we can explicitly calculate the normalization factor

$$\begin{aligned} N^{-1} &= \int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}} \\ &= \int \mathcal{D}\phi \exp\left\{\frac{i}{2}\int d^4x\phi(-\partial^2-m^2+i\epsilon)\phi\right\} = \frac{1}{\sqrt{\det A}}\,. \end{aligned} \tag{1.93}$$

Then, it is easy to obtain

$$Z[J] = \frac{\sqrt{\det A}}{\sqrt{\det A}}\exp\left\{-\frac{1}{2}\int d^4x\int d^4y J(x)D_F(x-y)J(y)\right\}\,, \tag{1.94}$$

which coincides with (1.87) previously obtained with another method.

Notice

1) What is $\det A$, where $\det A = \det(-\partial^2 - m^2 + i\epsilon)$? It is an example of a functional determinant, and A here is a partial differential operator, which is the kernel of bilinear form that entered the action.

   We can study the eigenvalues of this operator for example in momentum space in Euclidean formulation. We have:

   $$\det A \propto \prod_{k_E}(k_E^2 + m^2) \tag{1.95}$$

   with $k_E^2 = k_4^2 + \mathbf{k}^2$ and $k_0 = ik_4$; this is a divergent quantity. However since this is a quantity that does not depend on J, we can cancel it by appropriately choosing the normalization as we have already done.

2) Eq. (1.94) shows that for the scalar free theory there is only one connected [1] Green's function: $D_F(x-y) = G^{(2)}(x-y)$. This is the Feynman propagator, it is represented in a Feynman diagram with a line. In general:

   $$Z[J] = e^{\frac{i}{\hbar}W[J]}\,,\quad W[J] = -i(\hbar)\ln Z[J] \tag{1.96}$$

   and

   $$G^{(n)}_{\text{conn}}(x_1,\dots,x_n) = (-i)^{n-1}\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,, \tag{1.97}$$

   with

   $$W[J] = (\hbar)\sum_{n=0}^{\infty}\frac{(i)^{n-1}}{n!}\int d^4x\dots d^4x_n G^{(n)}_{\text{conn}}(x_1,\dots,x_n)\,J(x_1)\dots J(x_n)\,. \tag{1.98}$$

   W[J] is the generating functional of the connected Green's functions.

---

[1] Connected Green's functions can be represented by connected diagrams, *i.e.*, diagrams such that one can go throughout the diagram between any given two points of it.

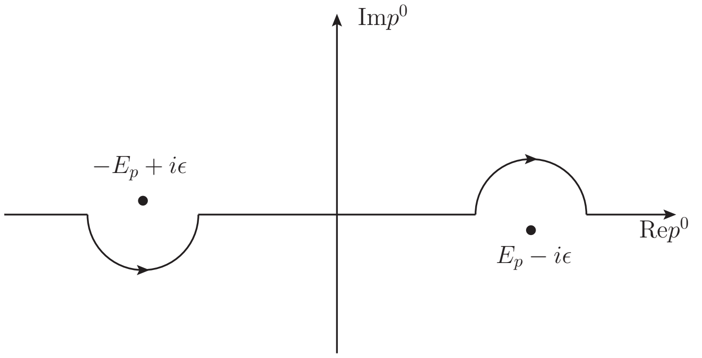


Figure 1.4: Poles of the free propagator.

3) The Feynman propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{1.99}$$

has been extensively studied in notes of the course *Relativity, Particles and Fields*, it also reads

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = G^{(2)}(x-y)\,. \tag{1.100}$$

We work on (1.99) to show it can be written as (1.100). The poles are

$$p_0^2 = \mathbf{p}^2 + m^2 - i\epsilon \;\Rightarrow\; p_0 = \pm E_p \mp i\epsilon\,, \tag{1.101}$$

which are displayed in Fig. 1.4.

For $x_0 > y_0$ we can close the contour in the lower half plane taking the contribution of the pole $E_{\mathbf{p}} - i\epsilon$. On the other hand, if $x_0 < y_0$ we can close the contour in the upper half plane taking the contribution of the pole $-E_{\mathbf{p}} + i\epsilon$. So we find

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip\cdot(x-y)}}{(p_0 - E_{\mathbf{p}} + i\epsilon)(p_0 + E_{\mathbf{p}} - i\epsilon)} \\
&\quad = -(2\pi i) \int \frac{d^3\mathbf{p}}{(2\pi)^4} \frac{i}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}\Big|_{p_0 = E_{\mathbf{p}}} \theta(x_0 - y_0) \\
&\qquad + (2\pi i) \int \frac{d^3\mathbf{p}}{(2\pi)^4} \frac{i}{-2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}\Big|_{p_0 = -E_{\mathbf{p}}} \theta(y_0 - x_0)\,,
\end{aligned} \tag{1.102}$$

where the minus sign in the first term of the right hand side of the equation arises because the corresponding contour integral is clockwise. Therefor for $x_0 > y_0$

$$\begin{aligned}
D_F(x-y) &= -\frac{2\pi i}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \\
&= \langle 0|\phi(x)\phi(y)|0\rangle\,,
\end{aligned} \tag{1.103}$$

(For the last equality please see chapter 7 of notes of the course *Relativity, Particles and Fields.*) and for $x_0 < y_0$

$$D_F(x-y) = \frac{2\pi i}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i}{-2E_{\mathbf{p}}} e^{ip_0(x_0-y_0) - i\mathbf{p}(\mathbf{x}-\mathbf{y})} = + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(y-x)}$$

$$= \langle 0|\phi(y)\phi(x)|0\rangle \,, \tag{1.104}$$

so that

$$\begin{aligned} &D_F(x-y) = \theta(x_0-y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0-x_0)\langle 0|\phi(y)\phi(0)|0\rangle \\ &\equiv \langle 0|T\phi(x)\phi(y)|0\rangle \,. \end{aligned} \tag{1.105}$$

2 Perturbation theory

2.1 Introduction

We consider the free scalar case. We can now prove the Wick's theorem [2] in the functional integration framework. Let us start with

$$G^{(n)}(x_1,\, \dots\, , x_n) = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots J(x_n)} Z[J]\Big|_{J=0} \,. \tag{2.1}$$

Then

$$\begin{aligned} G^{(2)}(x_1, x_2) &= (-i)^2 \frac{\delta^2}{\delta J(x_1)\delta J(x_2)} \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y)\right\}\Big|_{J=0} \\ &= D_F(x_1 - x_2)\,. \end{aligned} \tag{2.2}$$

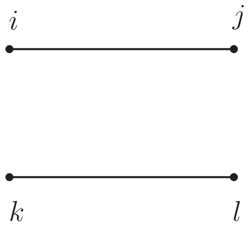
Then one can notice immediately that in this case all the Green's function with an odd number of points are zero.
Let us calculate the 4-point Green's function:

$$G^{(4)}(x_1, x_2, x_3, x_4) = (-i)^4 \frac{\delta^4}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \left(-\frac{1}{2}\right)^2 \frac{1}{2!} \sum_{i,j,k,\ell} J_i J_j J_k J_\ell D^F_{ij} D^F_{k\ell}\,, \tag{2.3}$$

where we have listed the only term in the series expansion of $Z[J]$ that gives a nonzero result and we find convenient to convert the integrals in the exponent of $Z[J]$ into sums over discrete points, with notations

$$J_i = J(x_i)\,, \quad D^F_{ij} = D_F(x_i - x_j)\,. \tag{2.4}$$

The structure is conveniently expressed in terms of diagrams of the type:



$$= \quad D^F_{ij}\; D^F_{kl} \quad . \tag{2.5}$$

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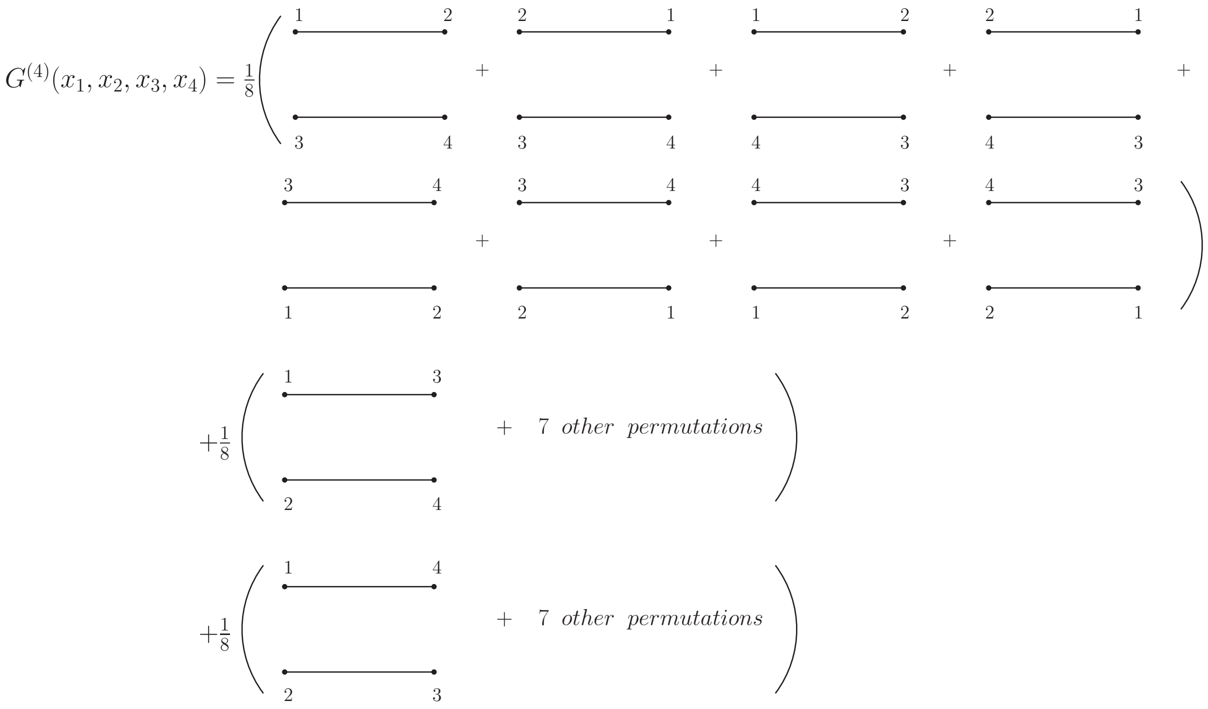
[2] See also section 10.5 of notes of the course *Relativity, Particles and Fields.*

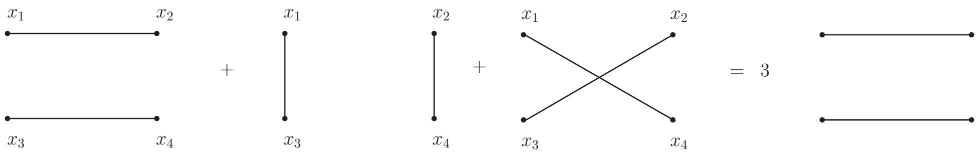
The derivative produce $4 \cdot 3 \cdot 2 = 24$ terms with the structure that can be represented diagrammatically as



or equivalently

$$\begin{aligned} G^{(4)}(x_1,x_2,x_3,x_4) = D_F(x_1-x_2)D_F(x_3-x_4) \;&+\; D_F(x_1-x_3)D_F(x_2-x_4) \\ &+\; D_F(x_1-x_4)D_F(x_2-x_3)\,, \end{aligned} \tag{2.6}$$

or since these diagrams are topologically equivalent we can also represent them as follows (Feynman diagrams in position space)



(2.7)

Eq. (2.6) is precisely the content of the Wick theorem that we have seen in the canonical quantization framework last semester. Indeed for the four-point function we have

$$\begin{aligned} &T\left\{\phi_1\phi_2\phi_3\phi_4\right\} =: \phi_1\phi_2\phi_3\phi_4 : + D_{12} : \phi_3\phi_4 : + \\ &D_{13} : \phi_2\phi_4 : + D_{14} : \phi_2\phi_3 : + D_{23} : \phi_1\phi_4 : + \\ &D_{24} : \phi_1\phi_3 : + D_{34} : \phi_1\phi_2 : + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}\,, \end{aligned} \tag{2.8}$$

so that

$$\langle 0|T\left\{\phi_1\phi_2\phi_3\phi_4\right\}|0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}\,. \tag{2.9}$$

Momentum space Feynman rules

We can Fourier transform the source $J(x)$, then we have

$$J(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{J}(k)\,. \tag{2.10}$$

Recalling Eq. (1.89), the exponent of (free) $Z[J]$, which has a quadratic form, can be transformed into the Fourier space as:

$$\begin{aligned}
&-\frac{1}{2}\int d^4x \int d^4y J(x) D_F(x-y) J(y) \\
&= -\frac{1}{2}\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\int d^4x \int d^4y\, i \frac{\tilde{J}(p_1) e^{-i(p_1+k)\cdot x} e^{-i(p_2-k)\cdot y}\tilde{J}(p_2)}{k^2-m^2+i\epsilon} \\
&= -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\,,
\end{aligned} \tag{2.11}$$

where we used

$$\int \frac{d^4x}{(2\pi)^4} e^{-i(p+k)\cdot x} = \delta^4(p+k) \tag{2.12}$$

to evaluate the the integrals with respect to x and y, and in turn to get rid of the integrals with respect to p_1 and p_2.

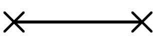
So we can write for the scalar free case

$$Z_{\text{free}}[J] = 1 - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon} + \frac{1}{2!}\left[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\right]^2 + \cdots\,,$$

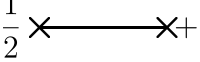
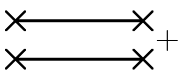
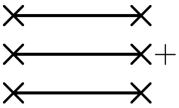
with the corresponding Feynman diagrams (free propagation, source, propagation between two sources respectively) as follows

 $= \dfrac{i}{k^2-m^2+i\epsilon}$

J  $= iJ(k)$

J  $J \;=\; -i\displaystyle\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+i\epsilon}\,.$ (2.13)

So that we can write graphically the functional generator for the free scalar case:

$Z[J] = 1 + \dfrac{1}{2}$  $+\, \dfrac{1}{2!}\left(\dfrac{1}{2}\right)^2$  $+\, \dfrac{1}{3!}\left(\dfrac{1}{2}\right)^3$  $+\cdots$ (2.14)

2.2 Perturbation theory: interacting case $\lambda\phi^4$

Let us recall the Lagrangian for the $\lambda\phi^4$ interacting theory that reads

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4\,, \tag{2.15}$$

and we want to calculate Green's functions in this theory.
The generator functional is now

$$Z[J] = N \int \mathcal{D}\phi \, e^{i \int d^4x (\mathcal{L} + J\phi)} \,, \tag{2.16}$$

normalized so that $Z[0] = 1$ and with $\mathcal{L}$ given by (2.15). In this case we can no longer calculate exactly $Z[J]$. To calculate the Green's functions we have to use a perturbative expansion defined in terms of the interaction term:

$$S_I = S - S_0 \,, \quad S_0 = S_{\text{free}} \,. \tag{2.17}$$

S_I is regarded as small; typically the interaction term is weighted by a small coupling constant like the λ in $\lambda\phi^4/4!$. Then we can write the identity:

$$\begin{aligned} &\int \mathcal{D}\phi \, exp \left\{ i \left(S_0[\phi] + S_I[\phi] + \int d^4x J(x)\phi(x) \right) \right\} \\ &\equiv \exp \left\{ iS_I \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} \int \mathcal{D}\phi \, \exp \left\{ i \left(S_0[\phi] + \int d^4x J(x)\phi(x) \right) \right\} \\ &= \exp \left\{ iS_I \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} Z_0[J] \,, \end{aligned} \tag{2.18}$$

where $Z_0[J] = Z_{\text{free}}[J]$ is the functional generator for the free case that we are able to calculate in a closed form (we are going to use $Z_{\text{free}}[J]$ in the following).

The perturbative series is generated by expanding the exponential factor $\exp(iS_I[\delta/i\delta J])$ in powers of S_I and performing the functional derivatives as indicated.
This is equivalent to expanding $\exp(iS_I[\phi])$ under the path integral. So we obtain in perturbation theory this general formula for the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(iS_I[\phi] \right)^m \right] e^{iS_0[\phi]}}{\int \mathcal{D}\phi \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(iS_I[\phi] \right)^m \right] e^{iS_0[\phi]}} \,. \tag{2.19}$$

Let us now summarize the interaction Lagrangian, action, and the generating functional for the $\lambda\phi^4/4!$ theory as follows:

$$\begin{cases} \mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 \,, \\ S_I = \int d^4x \left[-\frac{\lambda}{4!}\phi^4 \right] \,, \end{cases} \tag{2.20}$$

$$\begin{cases} Z[J] = N \, \exp \left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} Z_{\text{free}}[J] \,, \\ \frac{1}{N} = \exp \left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} Z_{\text{free}}[J] \mid_{J=0} \,, \\ Z_{\text{free}}[J] = \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \right\} \,. \end{cases} \tag{2.21}$$

We consider a perturbative expansion in a small coupling constant λ so that

$$\exp \left\{ -i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 \right\} \simeq 1 - i\frac{\lambda}{4!} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^4 + \cdots \tag{2.22}$$

Let us calculate the following quantities

$$\frac{\delta}{i\delta J(y)} Z_{\text{free}}[J] = \left\{+i\int d^4x D_F(x-y)J(x)\right\} Z_{\text{free}}[J]\,, \tag{2.23}$$

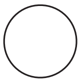
$$\left(\frac{\delta}{i\delta J(y)}\right)^2 Z_{\text{free}}[J] = \left\{D_F(0) - \left(\int d^4x D_F(x-y)J(x)\right)^2\right\} Z_{\text{free}}[J]\,, \tag{2.24}$$

$$\begin{aligned}
&\left(\frac{\delta}{i\delta J(y)}\right)^3 Z_{\text{free}}[J] = \left\{iD_F(0)\left(\int d^4x D_F(x-y)J(x)\right)\right.\\
&\left.+2iD_F(0)\left(\int d^4x D_F(x-y)J(x)\right) - i\left(\int d^4x D_F(x-y)J(x)\right)^3\right\} Z_{\text{free}}[J]\\
&= \left\{3iD_F(0)\left(\int d^4x D_F(x-y)J(x)\right) - i\left(\int d^4x D_F(x-y)J(x)\right)^3\right\} Z_{\text{free}}[J]\,,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&\left(\frac{\delta}{i\delta J(y)}\right)^4 Z_{\text{free}}[J] = \left\{3(D_F(0))^2 - 6D_F(0)\left(\int d^4x D_F(x-y)J(x)\right)^2\right.\\
&\left.+\left(\int d^4x D_F(x-y)J(x)\right)^4\right\} Z_{\text{free}}[J]\,.
\end{aligned} \tag{2.26}$$

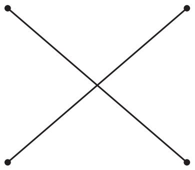
We can use Feynman diagrams in position space to represent those formulas and we list them as follows

$D_F(x-y)$



$D_F(0)$

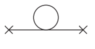
$J(x)$



$-i\lambda$

.

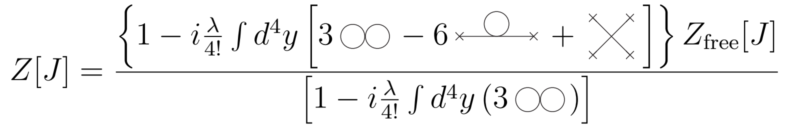
Then we can write

$\lambda\left(\frac{\delta}{i\delta J(y)}\right)^4 Z_{\text{free}}[J] = \lambda\Big\{3$  $-\,6$  $+$  $\Big\} Z_{\text{free}}[J]\,.$ (2.27)

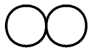
Then we can calculate also at this same order the normalization factor

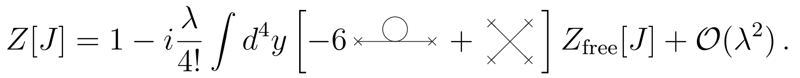
$$
\begin{aligned}
N^{-1} &= \exp\left\{-i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4\right\} Z_{\text{free}}[J]\Bigg|_{J=0} \\
&= \left(1 - i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right) Z_{\text{free}}[J]\Bigg|_{J=0} \\
&= 1 - i\frac{\lambda}{4!}\int d^4y\, 3(D_F(0))^2\,, \qquad (2.28)
\end{aligned}
$$

where in the last step we used (2.27) and put $J = 0$. So at the first order in the λ expansion we have



(2.29)

We see already that the normalization N contains vacuum diagrams that cancel with the same vacuum diagrams contribution in $Z[J]$, at this order it is the diagram .
Then since the vacuum diagrams (vacuum fluctuations) cancel between the numerator and the denominator we are left with



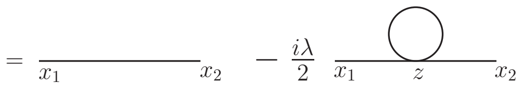
(2.30)

The cancellation of the vacuum diagrams occurs at any order in perturbation theory and it is a property of the normalized $Z[J]$.

Calculation of Green's functions in the $\lambda\phi^4/4!$ theory at order λ: $G^{(2)}$ and $G^{(4)}$

Let us start with the two point Green's function

$$
\begin{aligned}
G^{(2)}(x_1, x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J(x_1)\delta J(x_2)} Z[J]\Bigg|_{J=0} \\
&= (-1)\frac{\delta}{\delta J(x_1)\delta J(x_2)}\left[\left(1 - i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta}{i\delta J(z)}\right)^4\right) Z_{\text{free}}[J]\right]_{J=0} \\
&= D_F(x_1 - x_2) - i\frac{\lambda}{4!}2\cdot 6\, D_F(0)\int d^4z D_F(z - x_1)D_F(z - x_2) + \mathcal{O}(\lambda^2)
\end{aligned}
$$



. (2.31)

Up to order λ the effect of the interaction on the free particle propagator is described by the last diagram in (2.31). But what is the physical result of this interaction? We see now that the effect of the interaction is to change the value of the physical mass away from m.
We know that

$$D_F(x - y) = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \qquad (2.32)$$

has poles at $p^2 = \pm m$. Now let us evaluate the following quantity:

$$
\begin{aligned}
&-i\frac{\lambda}{2}D_F(0)\int d^4z D_F(z-x_1)D_F(z-x_2)\\
&= -i\frac{\lambda}{2}D_F(0)\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int d^4z\frac{ie^{-ip_1\cdot(z-x_1)}}{p_1^2-m^2+i\epsilon}\frac{ie^{-ip_2\cdot(z-x_2)}}{p_2^2-m^2+i\epsilon}\\
&= +i\frac{\lambda}{2}D_F(0)\int\frac{d^4p_1}{(2\pi)^4}\int d^4p_2\delta^4(p_1+p_2)\frac{e^{ip_1\cdot x_1+ip_2\cdot x_2}}{(p_1^2-m^2+i\epsilon)(p_2^2-m^2+i\epsilon)}\\
&= +i\frac{\lambda}{2}D_F(0)\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2}\,,
\end{aligned}
\tag{2.33}
$$

where we called $p_1 = p$. Therefore we obtain for the Green's function:

$$
G^{(2)}(x_1,x_2) = i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\epsilon}\left(1+\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}\right)+\mathcal{O}(\lambda^2)\,. \tag{2.34}
$$

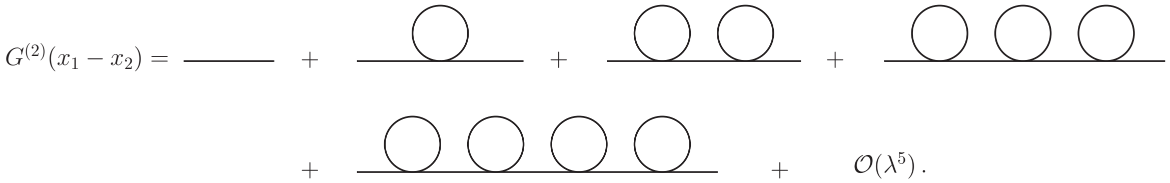
Formally we can rewrite

$$
1+\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}+\mathcal{O}(\lambda^2) = \left(\frac{1}{1-\frac{\lambda}{2}\frac{D_F(0)}{p^2-m^2+i\epsilon}}\right)\,, \tag{2.35}
$$

the last equality is valid up to $\mathcal{O}(\lambda^2)$. We have used

$$
\frac{1}{1-x} = 1+x+\mathcal{O}(x^2)\,, \tag{2.36}
$$

and this series resummation corresponds to taking into account all the diagrams of the *tadpole* type



Using (2.34) and (2.35) we obtain

$$
\begin{aligned}
G^{(2)}(x_1,x_2) &= i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\epsilon}\left(\frac{p^2-m^2+i\epsilon}{p^2-m^2+i\epsilon-\frac{\lambda}{2}D_F(0)}\right)\\
&= i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2-\frac{\lambda}{2}D_F(0)+i\epsilon}\,.
\end{aligned}
\tag{2.37}
$$

So we see that the Fourier transform of $G^{(2)}(x_1,x_2)$ now has a pole at $p^2 = m^2 + (\lambda/2)D_F(0) \equiv m^2 + \delta m^2 = m_R^2$. Hence *the particle that propagates in an interacting theory* has a physical mass equal to m_R (renormalized) which is the mass that can be measured. The mass m in the Lagrangian is not the same as the physical mass m_R in an interacting theory, and m can not be measured. The change in mass is related to the self-interacting term in the Lagrangian, namely $(\lambda/4!)\phi^4$.
The change in the mass, δm^2, is quadratically divergent:

$$
D_F(0) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} \sim \frac{p^4}{p^2}\,, \tag{2.38}
$$

indeed there are four powers of p in the numerator and two in the denominator. One can also see it by putting a cutoff in the above integral. The mass is changed by an infinite constant. The divergence is UV and it comes from very large momenta or very small space distances. We will deal with those infinities in the renormalization of $(\lambda/4!)\phi^4$ theory.

Notice: Power counting.
A way to evaluate the degree of divergence of the integrals of the type in eq. (2.38) is the power counting, i.e. to count the powers in p at the numerator and denominator and subtract them. In the case of eq. (2.38) at the numerator the momentum appears with exponent 4 and at the denominator appears with exponent 2, so that the divergence is $D = 4 - 2 = 2$ (quadratic divergence).

Let us now discuss the four point Green's function.

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) &= \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \\ &= \left.\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}, \end{aligned} \tag{2.39}$$

for the free case we have already obtain it in eq. (2.6), that we repeat here

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) = D_F(x_1 - x_2)D_F(x_3 - x_4) &+ D_F(x_1 - x_3)D_F(x_2 - x_4) \\ &+ D_F(x_1 - x_4)D_F(x_2 - x_3) \,. \end{aligned} \tag{2.40}$$

We are interested only in the order λ of th expansion of $Z[J]$:

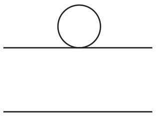
$$\begin{aligned} Z[J] = N &\left[1 - i\frac{\lambda}{4!}\int d^4z \left(\frac{\delta}{i\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right] \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x)D_F(x-y)J(y)\right\} \\ = N &\left[1 - \int d^4z \left(\frac{i\lambda}{4!}\right) 3(D_F(0))^2 + \frac{i\lambda}{4!} 6D_F(0)\int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^2 \right. \\ &\left. -\frac{i\lambda}{4!}\int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^4\right] \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x)D_F(x-y)J(y)\right\} \\ &+\mathcal{O}(\lambda^2)\,. \end{aligned} \tag{2.41}$$

Note that the above relation is identical to eq. 2.30 after substituting the normalization factor N. Now, one can see that:

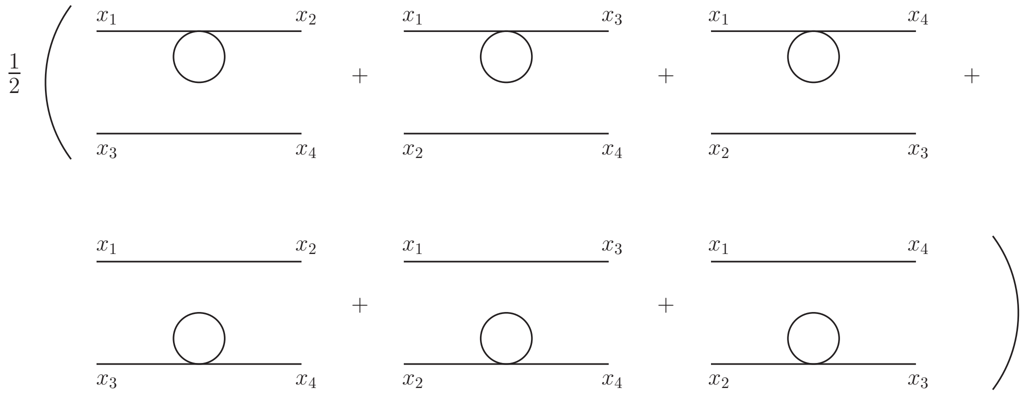
- The 1 in eq. (2.41) gives the terms in (2.40) when one applies the functional derivative with the four currents;
- the second term, in principle, gives a multiple of the vacuum diagram, but it does not contribute since it will be canceled out by the renormalization factor N;
- the third term in (2.41) gives, after derivation:

$$\begin{aligned} -i\frac{\lambda}{2}D_F(0)\int d^4z \quad &[D_F(z-x_1)D_F(z-x_2)D_F(x_3-x_4) \\ &+D_F(z-x_1)D_F(z-x_3)D_F(x_2-x_4) \\ &+D_F(z-x_1)D_F(z-x_4)D_F(x_2-x_3) \end{aligned}$$

$$
\begin{aligned}
&+D_F(z-x_2)D_F(z-x_3)D_F(x_1-x_4)\\
&+D_F(z-x_2)D_F(z-x_4)D_F(x_1-x_3)\\
&+\,D_F(z-x_3)D_F(z-x_4)D_F(x_1-x_2)]
\end{aligned}
$$

$= -3i\lambda \Bigg($  $\Bigg)$, (2.42)

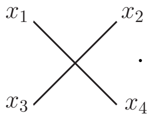
and the diagrams in the last line of (2.42) has to be understood as



, (2.43)

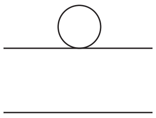
- the fourth term in (2.41) gives, after derivation:

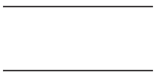
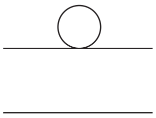
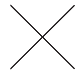
$$
-i\lambda\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)
$$

$= -i\lambda$ 

. (2.44)

We can then write:

$G^{(4)}(x_1,x_2,x_3,x_4) = 3\Big($  $\Big) - 3i\lambda\Bigg($  $\Bigg) - i\lambda\Big($  $\Big)$

$= 3\Big($  $\Big) - i\dfrac{\lambda}{4!}\Bigg[12\times 6\Bigg($  $\Bigg) + 24\Big($  $\Big)\Bigg]$. (2.45)

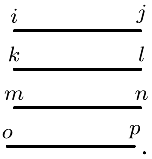
The numerical coefficients in (2.45) come from pure combinatorics and they are called weight factors. Let us see how it works. The contributing term in the $G^{(4)}$ is

$$
-i\frac{\lambda}{4!}\int d^4z\,\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}\left(\frac{\delta}{\delta J(z)}\right)^4 Z[J]_{\text{free}}\Bigg|_{J=0}. \qquad (2.46)
$$

The only non zero term comes from the term in the expansion of $Z[J]_{\text{free}}$ that contains four propagators:

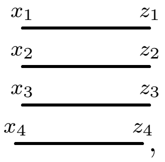
$$
\left(-\frac{1}{2}\right)^4\frac{1}{4!}\left[\int d^4x\int d^4yJ(x)D_F(x-y)J(y)\right]^4. \qquad (2.47)
$$

So at the end we will have four propagators

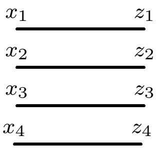
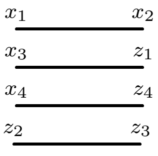
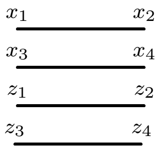


The differentiation over (2.47) provides all possible assignments of points x_1, x_2, x_3, x_4 and four points z (that it is convenient to split in z_1, z_2, z_3, z_4 and remembering at the end that $z_1 = z_2 = z_3 = z_4$) to the points i, j, k, l, m, n, o, p. Let us discuss the several contributions.
First let us assume that the pairs of points joined by the propagators are specified, e.g. $x_1 z_1$, $x_2 z_2$, $x_3 z_3$ and $x_4 z_4$. Then we have

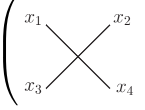
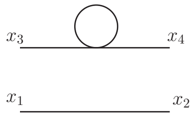
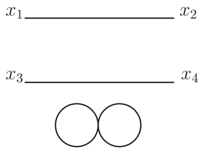
- an additional combinatorial factor following from the "horizontal" symmetry (exchange $x_1 \leftrightarrow z_1$) which is 2^n for n pairs,
- another combinatorial factor coming from the "vertical" symmetry, e.g. exchanging the pairs $x_1 - z_1$ with $x_2 - z_2$ in



which is $n!$ for n pairs. These factors cancel with the $1/(2^n n!)$ present in (2.47) and this happens for any term of the expansion contributing to the Green's functions. We have to remember the factor $1/N!$ that comes from the series expansion $exp(iS_I) = \sum_{N=0}^{\infty}(i)^N/N!(S_I)^N$. For the case $\mathcal{O}(\lambda)$ we have $N = 1$ and we should remember the factor $1/4!$ present in the definition of the interaction $\lambda/4!\phi^4$.
Finally we should take into account all the possible choices of pairs of points x_i and z_i joined by the propagator. One obtains

$4!$  $+ 3!\,4 \times 3$  $+ 3 \times 3$  .

Then for $z_1, z_2, z_3, z_4 \rightarrow z$ we get

$-i\frac{\lambda}{4!}\Bigg\{4!\Bigg($  $\Bigg) + 4! \times 3\Bigg($  $\Bigg) + 3 \times 3\Bigg($  $\Bigg)\Bigg\}$, (2.48)

We have three topologically distinct type of diagrams and we can understand the weight factors in front of each class in the following way:

1) − x_1 connected to z_1 or z_2 or z_3 or z_4: 4 ways,

- x_2 connected to the three z remained: 3 ways,
- x_3 connected to the two z remained: 2 ways,
- x_4 connected to the one z remained: 1 ways,

2)
- x_1 connected to x_2 or x_3 or x_4: 3 ways,
- z_i connected to the two x remained: 4 × 2 ways,
- remained three z_i connected to the one x remained: 3 ways,

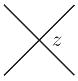
3)
- x_1 connected with x_2 or x_3 or x_4: 3 ways,
- z_1 connected with the three z remained: 3 ways.

Remember that the analytic expression for the first diagram (vertex diagram) in Eq. 2.48 is

$$-i\lambda \int d^4z D_F(x_1 - z) D_F(x_2 - z) D_F(x_3 - z) D_F(x_4 - z)\,. \qquad (2.49)$$

In summary, we can arrive at the following Feynman rules for $\lambda/4!\phi^4$ in coordinate space:

propagator line  $= \quad D_F(x-y)$

vertex  $= \quad -i\lambda$ with integration over z

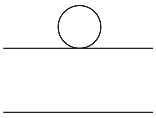
divide by the symmetry factor S (2.50)

It is important to remark here that dividing by the symmetry factor S is equivalent to multiplying by the weight of the diagram.

Notice

1) in the calculation of realistic processes involving e. g. electrons and photons, the particles are not identical and there are not symmetry factors.

2) the diagrams like



(2.51)

in $G^{(4)}$ only contribute to the trivial (diagonal) part of the S matrix so it is not relevant. It describes two particles moving independently and the effect of the interaction is to modify the propagator of one of them. This graph is called *disconnected.* The other graph of order λ,

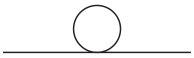
(2.52)

is *connected* (any line is connected to every line). Only connected Feynman diagrams contribute to the non-trivial part of the S matrix.

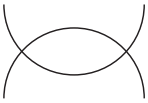
3) The symmetry factor has to be worked out for each graph. However in ϕ^4 a rule can be stated for *connected non-vacuum diagrams* (= graphs with external lines and no disjoint subgraphs). In such a graph, k internal lines are said to form an equivalent set if they all share the same vertices at both ends. If there are more than one such set containing respectively $k_1, k_2, ...$ internal lines, then the symmetry factor is

$$S = \prod_i k_i! \,. \tag{2.53}$$

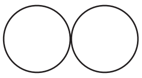
S here is the number of ways in which the lines and the vertices of the graphs can be rearranged without changing its connectivity.
For example we show some diagrams with the corresponding S:



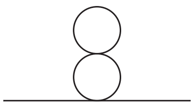
$S = 2$



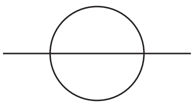
$S = 2$



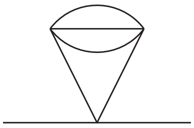
$S = 2 \times 2 \times 2$



$S = 2 \times 2$



$S = 3!$



$S = 3!\,2$

Let us briefly comment on the vacuum graph. At order λ we have

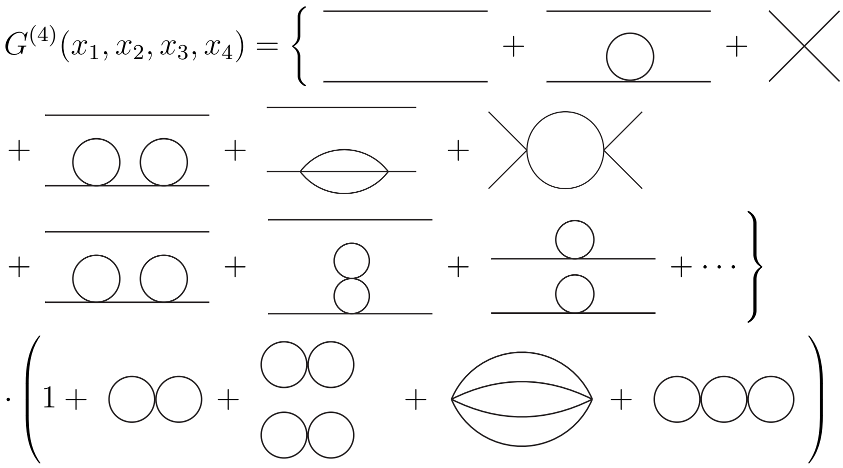
$$\bigcirc\!\!\bigcirc = i\frac{\lambda}{4!}\,3 \int d^4z (D_F(0))^2 \,. \tag{2.54}$$

This involves creation and annihilation of two virtual particle-antiparticle pairs. Leaving out the coefficient we rewrite the vacuum as

$$\begin{aligned} &-i\lambda \int d^4z \, \langle 0|T\phi(0)\phi(0)|0\rangle \langle 0|T\phi(0)\phi(0)|0\rangle \\ &= -i\lambda(2\pi)^4\delta^4(0) \left[\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\right]^2 \,. \end{aligned} \tag{2.55}$$

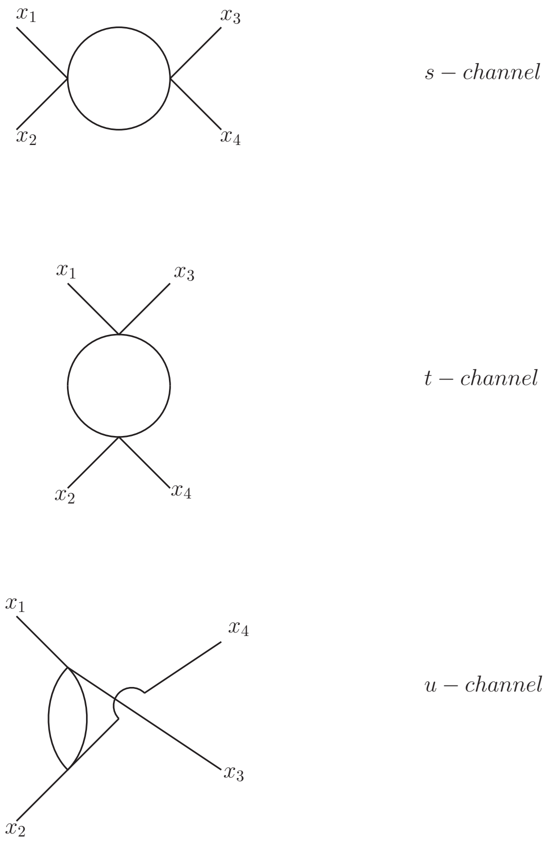
The factor $(2\pi)^4\delta^4(0)$ represents the integral $\int d^4z$ and should be interpreted as the total volume of space-time. Vacuum processes such this one occur *with uniform probability over all space-time* (the vacuum is not empty) and they can accompany any reaction we consider. This factor does not affect physical transition probabilities.

Let us finally present diagrammatically the four-point Green's function $G^{(4)}$ up to order λ^2:



(2.56)

where in the last line of the diagrammatic equation we have the vacuum to vacuum transition that are cancelled by the normalization. At order λ^2 we have three types of connected diagrams:



(2.57)

where the *u-channel* diagram is a non-planar diagram.

2.3 Connected Green's function

The Green's functions of order n are given by the sum of all diagrams with n external legs including the disconnected diagrams but excluding the vacuum diagrams that are

cancelled by the normalization factor. Disconnected graphs do not require separate calculations → they are products of lower order connected graphs.
The connected Green's functions are defined as the connected part of $G^{(n)}$: they are obtained by the sum of connected Feynman diagrams. It is typically simpler to work with the connected Green's functions $G^{(n)}_{\text{conn}}(x_1, \dots, x_n)$. Their generating functional is $W[J]$:

$$Z[J] = e^{iW[J]}\,, \quad W[J] = -i \ln Z[J]\,, \tag{2.58}$$

and the connected part of $G^{(n)}$ reads

$$G^{(n)}_{\text{conn}}(x_1, \dots, x_n) = \frac{1}{(i)^n} i \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots J(x_n)} \right|_{J=0} = \frac{1}{(i)^{n-1}} \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots J(x_n)} \right|_{J=0} . \tag{2.59}$$

Now we aim to calculate $W[J]$ using the expression found for $Z[J]$ at order λ in (2.41), which is:

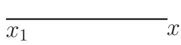
$$\begin{aligned} Z[J] = N &\left[1 - \int d^4z \left(\frac{i\lambda}{4!} \right) 3(D_F(0))^2 + \frac{i\lambda}{4!} 6 D_F(0) \int d^4z \left(\int d^4z' D_F(z-z') J(z') \right)^2 \right. \\ &\left. -\frac{i\lambda}{4!} \int d^4z \left(\int d^4z' D_F(z-z') J(z') \right)^4 \right] \exp\left\{ -\frac{1}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \right\} \\ &+ \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{2.60}$$

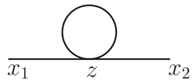
We obtain that

$$\begin{aligned} iW[J] = \ln N &- \frac{1}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) - \frac{i\lambda}{4!} \int d^4z\, 3(D_F(0))^2 \\ &+ \frac{i\lambda}{4!} 6 D_F(0) \int d^4z \left(\int d^4z' D_F(z-z') J(z') \right)^2 - \frac{i\lambda}{4!} \int d^4z \left(\int d^4z' D_F(z-z') J(z') \right)^4 \\ &+ \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.61}$$

and applying (2.59) to (2.61) we obtain

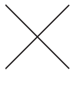
$$G^{(2)}(x_1, x_2) = G^{(2)}_{\text{conn}}(x_1, x_2) = D_F(x_1 - x_2) - i\frac{\lambda}{2} \int d^4z D_F(x_1 - z) D_F(0) D_F(z - x_2)$$

$=$ 

$- \frac{i\lambda}{2}$ 

, (2.62)

$$G^{(4)}_{\text{conn}}(x_1, x_2, x_3, x_4) = -i\lambda \int d^4z D_F(x_1 - z) D_F(x_2 - z) D_F(x_3 - z) D_F(x_4 - z)$$

$= -i\lambda$  (2.63)

So we see that $G^{(2)}$ is connected as in the free case but now the interaction generates new connected terms. In the free case all connected Green's functions for $n > 2$ were zero. When there is an interaction $G^{(n)}_{\text{conn}}(x_1, \dots, x_n) \neq 0$ for $n > 2$.

Momentum space

Propagators have their simplest form in momentum space. For this reason one typically evaluates the Fourier transform of the Green's functions we have calculated.
The Fourier transform of the Green's functions is defined as:

$$\tilde{G}^{(n)}(p_1, \dots, p_n)(2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) =$$

$$\int d^4x_1 \dots d^4x_n e^{+ip_1\cdot x_1+\dots+ip_n\cdot x_n} G^{(n)}(x_1,\,\dots,x_n)\,, \tag{2.64}$$

where the Dirac delta function occurs because translation invariance implies that $G^{(n)}(x_1,\,\dots,x_n)$ depends only on the difference of the x_i. Eq. (2.64) corresponds to

$$\begin{aligned}&G^{(n)}(x_1,\,\dots,x_n) = \\ &\int \frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4} e^{-ip_1\cdot x_1-\dots-p_n\cdot x_n}(2\pi)^4\delta^4(p_1+p_2+\,\dots\,+p_n)\tilde{G}^{(n)}(p_1,\,\dots,p_n)\,.\end{aligned} \tag{2.65}$$

All the momenta in (2.64) are treated as outgoing momenta $e^{ip\cdot x}$. Due to the $i\epsilon$ prescription in the propagator, positive particle frequencies propagate into the future $x_n^0 \to \infty$. In actual physical transitions

$$\begin{cases} x_1^0, \dots, x_k^0 \to -\infty\,, \\ x_{k+1}^0, \dots, x_n^0 \to +\infty\,, \end{cases} \tag{2.66}$$

and therefore the momenta $p_1, \dots, p_n$ defined in (2.64) are related to the physical momenta $\tilde{p}_i(\tilde{p}_i^0>0)$ of the initial and final particles as follows:

$$\begin{cases} p_i = -\tilde{p}_i\,, & i = 1, \dots, k \\ p_i = \tilde{p}_i\,, & i = k+1, \dots, n\,. \end{cases} \tag{2.67}$$

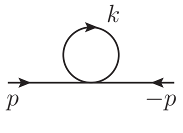
Therefore we have

$$\begin{aligned}&(2\pi)^4\delta^4(p_1+p_2)\tilde{G}^{(2)}(p_1,p_2) = \int d^4x_1\int d^4x_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2}G^{(2)}(x_1,x_2)\\ &= \int d^4x_1\int d^4x_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2}\int\frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon}\\ &= \int\frac{d^4k}{(2\pi)^4}\delta^4(p_1-k)\delta^4(p_2+k)(2\pi)^4(2\pi)^4\frac{i}{k^2-m^2+i\epsilon}\\ &= (2\pi)^4\delta^4(p_1+p_2)\frac{i}{p_1^2-m^2+i\epsilon}\\ &\Rightarrow \tilde{G}^{(2)}(p_1,p_2) \equiv G^{(2)}(p_1,-p_1) = \frac{i}{p_1^2-m^2+i\epsilon}\,,\end{aligned} \tag{2.68}$$

and we can calculate the Fourier transform of (2.63) to get

$$\begin{aligned}G^{(2)}(p,-p) &\equiv G^{(2)}(p)\\ &= \frac{i}{p^2-m^2+i\epsilon} - i\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{p^2-m^2+i\epsilon}+\mathcal{O}(\lambda^2)\end{aligned}$$

$=$ 

$+\frac{1}{2}$ 

$+\mathcal{O}(\lambda^2)$ (2.69)

Feynman rules for $\lambda\phi^4$

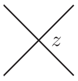
Position space Feynman rules

These are the rules that allow to get the analytic expression for each piece of a Feynman diagram. So the position space Feynman rules read

- for each propagator one has

$$= D_F(x-y)\,, \tag{2.70}$$

- for each vertex one has



$$= (-i\lambda)\int d^4z\,, \tag{2.71}$$

- for each external point one has

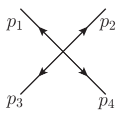
$$= 1 \tag{2.72}$$

- a loop diagram may have some left-over symmetry factor if there are exchanges of internal propagators and vertices that leave the diagram unchanged. In this case we have to divide the value of the diagram by the symmetry factor associated with the exchange of internal propagators and vertices.

The factor $(-i\lambda)$ can be thought as the amplitude for the emission and/or absorption of particles at a vertex. The integral $\int d^4z$ tells us that we have to sum over all points where this process can occur → this is the superposition principle of QM → when a process can happen in different ways we add the amplitude for each possibility. Moreover in order to calculate each individual amplitude, the Feynman rules tell us to multiply the amplitudes (propagators and vertices) for each of the independent part of the process.

Momentum space Feynman rules

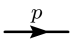
By Fourier transform one can go to momentum space. Then to the propagators one assigns a 4-momentum p, indicating in general the direction of the momentum with an arrow (since in this theory $D_F(x-y) = D_F(y-x)$ the direction of p in the individual propagator line is arbitrary).
The vertex becomes



$$\Leftrightarrow -i\lambda \int d^4z e^{+i(p_1+p_2+p_3+p_4)\cdot z} = -i\lambda\,(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\,. \tag{2.73}$$

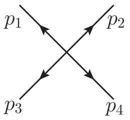
The Feynman rules are:

1) for the propagator (the arrow is arbitrary)



$$= \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{2.74}$$

2) for the vertex



$$= -i\lambda\,, \tag{2.75}$$

3) for the external point

$$= e^{-ipx} \tag{2.76}$$

4) divide by the symmetry factor,

5) impose energy momentum conservation at each vertex,

6) integrate over each undetermined momentum (loop) with a weight

$$\int \frac{d^4k}{(2\pi)^4} \,. \tag{2.77}$$

In order to get the contribution to $\tilde{G}^{(n)}(p_1, \ldots, p_n)$ draw all possible arrangements which are topologically inequivalent after having identified the external legs. The number of ways a given diagram can be drawn is the topological weight of the diagram $1/S$.
In a scalar theory with no derivative coupling the choice of the orientation of an internal line is irrelevant.
If V is the number of vertices and I the number of internal lines, there are $V-1$ conservation rules (coming from conservation of momentum at the vertices and the fact that a global momentum δ should factorize) and so at most there are $I-V+1$ non trivial integration or loops: $L = I - V + 1$.

Notice:
In relativistic QFT virtual transitions conserve both momentum and energy but the squared mass becomes unphysical: $p^2 \neq m^2$ in the propagator inside the loop. For example in the loop diagram in (2.69), recalling the expression as follows

$$-i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{p^2-m^2+i\epsilon}\,, \tag{2.78}$$

we notice that the loop momentum k^2 is not m^2, it is integrated over all k.

2.3.1 loop expansion

One can expand $W[J]$ in terms of the number of loops. This can be done by keep tracking the factor of $\hbar$ in our calculations. For this purpose we start from the functional integral defined in eq. 1.72, and expand it to all orders in perturbation theory. Let us determine the power of $\hbar$ in front of a *connected* Feynman diagram contributing to $W[J]$ for $\lambda\phi^4$. Each propagator has a factor of $\hbar$. Any vertex yields a power of $\hbar^{-1}$. In the same way at the end of all external lines is attached a factor of J which also yields a factor of $\hbar^{-1}$. Calling E the number of external lines (propagator joining a vertex to a source of J), we find that the power

$$\hbar^{I+E-(V+E)+1} \,, \tag{2.79}$$

the last factor of $\hbar$ has been added for our choice of normalization of $W[J]$. Now we use the relation $L = I - V + 1$ to conclude that the power of each diagram is $\hbar^L$; the power of $\hbar$ that we have found counts the number of loops of a diagram.

We have thus shown that the expansion in powers of $\hbar$ is a reordering of perturbation theory according to the number of loops of the Feynman diagrams.

2.4 One particle irreducible Green's functions or connected proper vertex functions

We will be dealing with several type of Feynman diagrams, for instance **connected** and also **amputated** diagrams. The integration associated with a Feynman diagram runs over loop momenta, it is convenient to introduce a reduced diagram which represents the loop integrations by removing all external lines. This is called an amputated diagrams, *i.e.*, without external lines. The external lines can be eliminated by multiplying by the reverse of the corresponding propagators.

The connected proper vertex function $\Gamma^{(n)}(x_1, \dots, x_n)$, which are also called one-particle-irreducible (1PI) Green's functions, are given by the amputated Feynman diagrams that are one-particle-irreducible, *i.e.*, they remain connected after that an arbitrary internal line is cut. In lowest order (*i.e.*, with no loop considered) the connected proper vertex functions coincide (when appropriately normalized) with the vertices of the original Lagrangian.

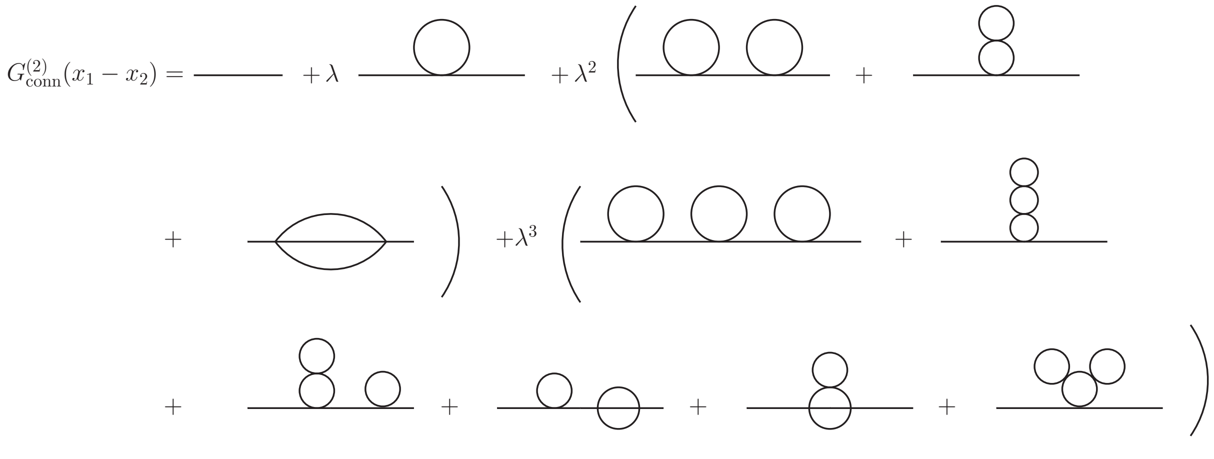
The connected proper vertex functions are the building blocks of perturbation theory. We can define the generating functional Γ of the $\Gamma^{(n)}(x_1, \dots, x_n)$:

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots \int d^4x_n \Gamma^{(n)}(x_1, \dots, x_n)\phi(x_1) \dots \phi(x_n)\,, \tag{2.80}$$

with

$$\Gamma^{(n)}(x_1, \dots, x_n) = \left.\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1) \dots \delta\phi(x_n)}\right|_{\phi=0}\,. \tag{2.81}$$

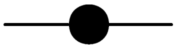
$\Gamma[\phi]$ is called *effective action.* We want to see how $\Gamma[\phi]$ is related to $W[J]$.
Before doing this let us comment about the role of 1PI Green's functions with a concrete example. We consider the two-point connected Green's function up to order λ^4. Schematically (without indicating the numerical factors or the i) we have



(2.82)

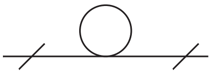
These are all connected diagrams and we would like to derive a method to sum them. The sum of all these diagrams is called complete or dressed propagator and is denoted as follows:

$$G^{(2),\mathrm{full}}_{\mathrm{conn}}(x_1 - x_2) = \qquad . \tag{2.83}$$



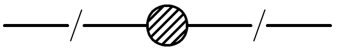
We have seen that the effect of order λ and higher orders is to change the physical mass away from the "bare mass" m and hence to give rise to a self-energy. The graphs above all contribute to the self-energy. The reason for selecting 1PI diagrams is that one particle reducible diagrams can be decomposed into 1PI diagrams without further loop integration and if we know how to take care of the 1PI divergencies then also those of the reducible diagrams will be handled → to make the theory finite we need to make finite only the 1PI diagrams.

We define amputated diagrams by multiplying the external legs by inverse propagators, diagrammatically

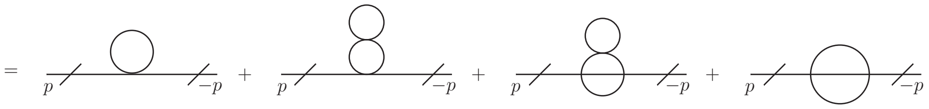


.

Then of the three graphs of order λ^2, the first is a product of lower order graphs but the other two are not. This is because the first graph contains a propagator → it is called a 1P reducible graph. This is not the case for the other two graphs which are 1PI. Based on this classification we can define *a proper self-energy part* as the sum of the amputated 1PI graphs:



$$= -i\Sigma(p) =$$

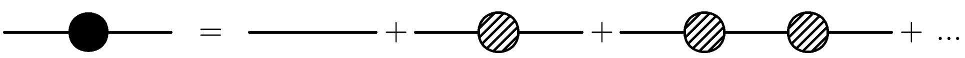


. (2.84)

The complete propagator in (2.82) may therefore be written in terms of the bare propagator $G_0(p) = i/(p^2 - m^2)$ and the proper self energy function $\sum(p)$ as follows:

$$\begin{aligned}
&G^{(2)}_{\mathrm{conn}}(p) = G_0(p) + G_0(p)\,(-i\Sigma(p))\,G_0(p) + G_0(p)\,(-i\Sigma(p))\,G_0(p)\,(-i\Sigma(p))\,G_0(p) + ...\\
&= G_0(p)\,(1 - i\Sigma(p)G_0(p) - i\Sigma(p)G_0(p)\,(-i\Sigma(p))\,G_0(p) + ...)\\
&= G_0(p)\,(1 + i\Sigma(p)G_0(p))^{-1} = \left[G_0^{-1}(p) + i\Sigma(p)\right]^{-1}\\
&= \frac{1}{\frac{p^2-m^2}{i} - \frac{\Sigma(p)}{i}} = \frac{i}{p^2 - m^2 - \Sigma(p)}\,,
\end{aligned} \tag{2.85}$$

or in diagrams:



(2.86)

Defining the physical mass by the pole of the complete propagator

$$G^{(2)}_{\mathrm{conn}}(p) = \frac{i}{p^2 - m^2_{\mathrm{phys}}}\,, \tag{2.87}$$

we obtain

$$m^2_{\mathrm{phys}} = m^2 + \Sigma(p)\,, \tag{2.88}$$

which justifies the name of "self-energy" that we have given to Σ. It represents the change in mass from the bare to the physical value calculated to all orders in perturbative theory. From (2.85) we have

$$\left(G^{(2)}_{\rm conn}(p)\right)^{-1} = G_0(p)^{-1} + i\Sigma(p)\,. \tag{2.89}$$

So the inverse of two-point function contains only 1PI graphs (apart from the inverse bare propagator). This is an example of proper vertex function that may be generalized. The two-point vertex function $\Gamma^{(2)}(p)$ is defined by

$$\begin{aligned} & G^{(2)}_{\rm conn}(p)\Gamma^{(2)}(p) = i \\ & \Rightarrow \Gamma^{(2)}(p) = p^2 - m^2 - \Sigma(p)\,, \end{aligned} \tag{2.90}$$

where in the last step we use the result of (2.85).

Now we want to obtain the generating functional for the $\Gamma^{(n)}$, and we will see the double correspondence: $\Gamma[\phi] \Leftrightarrow W[J]$ and $\phi \Leftrightarrow J$. To this aim we consider the vacuum expectation value of the field $\phi(x)$ in presence of an external source $J(x)$

$$\phi_{\rm cl}(x) = \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \tag{2.91}$$

We can derive eq. (2.91) as follows:

$$\begin{aligned} \phi_{\rm cl}(x) &= \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{1}{\hbar}\frac{\hbar}{i}\frac{\delta \ln Z[J]}{\delta J(x)} = \frac{1}{i}\frac{1}{Z}\frac{\delta Z[J]}{\delta J(x)} \\ &= \frac{1}{i}\frac{\int \mathcal{D}\phi\, i\phi(x) exp\left\{iS[\phi] + i\int d^4x J\phi\right\}}{\int \mathcal{D}\phi\, exp\left\{iS[\phi] + i\int d^4x J\phi\right\}} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \end{aligned} \tag{2.92}$$

The vacuum expectation value $\langle\phi\rangle = \langle 0|\phi|0\rangle / \langle 0|0\rangle$ is the limit of $\phi_{\rm cl}$ as $J(x) \to 0$. Here $\phi_{\rm cl}$ is determined by the external source J (it is a functional of the external source J). $\phi_{\rm cl}$ is an ordinary function and we can think of it as a classical field.

We now ask a question: what source J will produce a given function $\phi_{\rm cl}$? To answer we reformulate the problem so that it becomes convenient to use $\phi_{\rm cl}$ as independent variable in place of J. We define the "effective action" $\Gamma[\phi_{\rm cl}]$ by making a Legendre transformation

$$\Gamma[\phi_{\rm cl}] = W[J] - \hbar\int d^4x J(x)\phi_{\rm cl}(x)\,, \tag{2.93}$$

where J has to be eliminated in terms of $\phi_{\rm cl}$ by solving (2.91). $\Gamma[\phi_{\rm cl}]$ *is called effective action because it is a functional of the classical field* $\phi_{\rm cl}$ *and thus similar to* $S[\phi]$. The classical field $\phi_{\rm cl}$ minimizes the combination

$$\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\,, \tag{2.94}$$

or in other words is a solution of the "classical field" equations:

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = -\hbar J(x)\,. \tag{2.95}$$

(Notice that this equation is the reverse of eq. (2.91).) In fact by differentiating eq. (2.93) with respect to $J(x)$ we find

$$\frac{\delta W[J]}{\delta J(x)} = \int d^4y \, \underbrace{\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(y)}}_{-\hbar J(y)} \frac{\delta\phi_{\rm cl}(y)}{\delta J(x)} + \hbar\,\phi_{\rm cl}(x) + \hbar\int d^4y \frac{\delta\phi_{\rm cl}(y)}{\delta J(x)} J(y)\,, \tag{2.96}$$

where we used (2.95), and then we obtain

$$\phi_{\rm cl}(x) = \frac{1}{\hbar}\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}\,. \tag{2.97}$$

If $W[J]$ is known we can use (2.91) to determine $J(x)$ in terms of $\phi_{\rm cl}$ so that (2.93) can be written as a functional of $\phi_{\rm cl}$ which determines $\Gamma[\phi_{\rm cl}]$. The value in (2.91) when the external source is turned off ($J(x) = 0$) is the vacuum expectation value of $\phi(x)$ which by now we assume to be zero

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = 0\,. \tag{2.98}$$

The case of nonzero vacuum expectation value reflects spontaneous symmetry breaking and will be discussed later. Eq. (2.95) gives $J(x)$ in terms of $\phi_{\rm cl}(x)$ and in this sense is the inverse of (2.91). If we combine (2.95) and (2.98) we obtain

$$\left.\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}\right|_{\phi_{\rm cl}=0} = 0\,, \tag{2.99}$$

because when $J = 0$ it follows $\phi_{\rm cl} = 0$ and viceversa.
We can see then why $\Gamma^{(n)}$ is the n-point 1PI Green's function, *i.e*, the sum of all amputated 1PI Feynman diagram with n external legs. Let us consider

$$N\int \mathcal{D}\phi \, \exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\right]\right\}\,, \tag{2.100}$$

which is the generating functional of a new set of Green's functions corresponding to the new action $\Gamma[\phi]$. For $\hbar \to 0$ the integrand is dominated by the saddle point which occurs at $\phi = \phi_{\rm cl}$ due to

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_{\rm cl}} = -\hbar J(x)\,, \tag{2.101}$$

which is precisely the saddle point condition. Thus

$$\begin{aligned} & N\int \mathcal{D}\phi \, \exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi] + \hbar\int d^4x J(x)\phi(x)\right]\right\} \\ & \xrightarrow{\quad \hbar \to 0 \quad} \; N'\exp\left\{\frac{i}{\hbar}\left[\Gamma[\phi_{\rm cl}] + \hbar\int d^4x J(x)\phi_{\rm cl}(x)\right]\right\} \end{aligned} \tag{2.102}$$

and by (2.93) this is equal to

$$N'\exp(\frac{i}{\hbar}W[J])\,. \tag{2.103}$$

Therefore, the generating functional corresponding to $\Gamma[\phi]$ in the limit $\hbar \to 0$ is equal to $N'\exp(\frac{i}{\hbar}W[J])$. On the other hand, in the limit $\hbar \to 0$ only the tree level graphs of the new theory, whose action is $\Gamma[\phi]$, survive. Hence a vertex in one of these tree graphs is some $\Gamma^{(n)}$ by definition, whose form is given in (2.80). Moreover, recall that $W[J]$ is by definition the sum of all connected Green's function of the original theory and a connected Green's function can be dissected into 1PI components. So $\Gamma^{(n)}$ is a 1PI Green's function of the original theory. This will be explained more later.

Notice:
In the free case we have ($\hbar = 1$)

$$W[J] = -i \ln Z = +\frac{i}{2} \int d^4x \int d^4y J(x) D_F(x-y) J(y) \,, \tag{2.104}$$

and we obtain

$$\phi_{\text{cl}}(x) = \frac{\delta W[J]}{\delta J(x)} = i \int d^4x' D_F(x-x') J(x') \,, \tag{2.105}$$

so we see immediately that

$$\begin{aligned}(\partial^2 + m^2)\phi_{\text{cl}}(x) &= i \int d^4x' \underbrace{(\partial_x^2 + m^2) D_F(x-x')}_{-i\delta^4(x-x')} J(x') = \\ &= J(x) \,, \end{aligned} \tag{2.106}$$

which is identical to the classical field equation in presence of a source J and thus tells us why it is appropriate to refer to ϕ_{cl} as the classical field.
Moreover we can calculate $\Gamma[\phi_{\text{cl}}]$ explicitly as we know $W[J]$ and because (2.106) allows us to eliminate J in favour of ϕ_{cl}. Starting from

$$\Gamma[\phi_{\text{cl}}] = \frac{i}{2} \int d^4x \int d^4x' J(x) D_F(x-x') J(x') - \int d^4x J(x) \phi_{\text{cl}}(x) \tag{2.107}$$

and substituting $J(x)$ from (2.106) we obtain

$$\begin{aligned}\Gamma[\phi_{\text{cl}}] &= \frac{i}{2} \int d^4x \int d^4x' \left((\partial_x^2 + m^2)\phi_{\text{cl}}(x)\right) D_F(x-x') \left((\partial_{x'}^2 + m^2)\phi_{\text{cl}}(x')\right) \\ &- \int d^4x \phi_{\text{cl}}(x) (\partial_x^2 + m^2) \phi_{\text{cl}}(x) \\ &= \frac{1}{2} \int d^4x \phi_{\text{cl}}(x) (\partial_x^2 + m^2) \phi_{\text{cl}}(x) - \int d^4x \phi_{\text{cl}}(x) (\partial_x^2 + m^2) \phi_{\text{cl}}(x) \\ &= -\frac{1}{2} \int d^4x \phi_{\text{cl}}(x) (\partial_x^2 + m^2) \phi_{\text{cl}}(x) = \frac{1}{2} \int d^4x (\partial_\mu \phi_{\text{cl}} \partial^\mu \phi_{\text{cl}} - m^2 \phi_{\text{cl}}^2) \,, \end{aligned} \tag{2.108}$$

which is exactly the classical free-field theory action and justifies referring to $\Gamma[\phi_{\text{cl}}]$ as the effective action. Note that we have used integration by part and the relation $(\partial_x^2 + m^2) D_F(x-x') = -i\delta(x-x')$ to derive eq. (2.108). In the case of an interacting theory we will not be able to calculate ϕ_{cl} and $\Gamma[\phi_{\text{cl}}]$ exactly and there will be quantum correction to (2.106) and (2.108).
In the free theory we see, using eq. (2.108) and (2.81), that the only $\Gamma^{(n)} \neq 0$ is the $\Gamma^{(2)}$:

$$\Gamma^{(2)}(x, x') = -(\partial_{x'}^2 + m^2) \delta^4(x-x') \,, \tag{2.109}$$

with Fourier transform

$$(2\pi)^4 \delta^4(p_1 + p_2) \tilde{\Gamma}^{(2)}(p_1, p_2) = \int d^4x_1 \int d^4x_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} \Gamma(x_1, x_2)$$

$$
\begin{aligned}
&= \int d^4x_1 \int d^4x_2 e^{ip_1\cdot x_1 + ip_2\cdot x_2}(-1)(\partial^2_{x_1} + m^2)\delta^4(x_1 - x_2) \\
&= (-1)(-p_1^2 + m^2)(2\pi)^4\delta^4(p_1 + p_2) \\
&\Rightarrow \tilde{\Gamma}^{(2)}(p_1, p_2) = (p_1^2 - m^2)\,,
\end{aligned}
\tag{2.110}
$$

confirming that

$$
G^{(2)}_{\text{conn}}(p)\Gamma^{(2)}(p) = i\,. \tag{2.111}
$$

Notice that (2.93) is completely analogous to the thermodynamic equation $U = F + TS$ which gives the internal energy U as a function of the entropy S in terms of the free energy F regarded as a function of the temperature T.

Classical field and effective action at order λ

Using the expression for $W[J]$:

$$
\begin{aligned}
W[J] &= +i\ln N + \frac{i}{2}\int d^4x \int d^4y J(x)D_F(x-y)J(y) - \frac{\lambda}{4!}\int d^4z\, 3(D_F(0))^2 \\
&+ \frac{\lambda}{4!}6D_F(0)\int d^4z \left(\int d^4z' D_F(z-z')J(z')\right)^2 - \frac{\lambda}{4!}\int d^4z\left(\int d^4z' D_F(z-z')J(z')\right)^4\,,
\end{aligned}
\tag{2.112}
$$

and using the definition of classical field $(\delta W[J])/(\delta J)$ we get

$$
\begin{aligned}
\phi_{\text{cl}}(x) &= +i\int d^4y D_F(x-y)J(y) + \frac{1}{2}\lambda\int d^4z\, d^4z'\, D_F(x-z)D_F(z-z)D_F(z-z')J(z') \\
&-\frac{\lambda}{6}\int d^4z d^4z_1' d^4z_2' d^4z_3' D_F(x-z)D_F(z_1'-z)D_F(z_2'-z)D_F(z_3'-z)J(z_1')J(z_2')J(z_3') \\
&+\mathcal{O}(\lambda^2)\,,
\end{aligned}
\tag{2.113}
$$

and we obtain for the effective action

$$
\begin{aligned}
\Gamma[\phi_{\text{cl}}] &= W[J] - \int d^4 J(x)\phi_{\text{cl}}(x) \\
&= +i\ln N - \frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2)J(y_1)J(y_2) - \frac{1}{8}\lambda\int d^4x[D_F(0)]^2 \\
&-\frac{\lambda}{4}\int d^4x d^4y_1\, d^4y_2\, D_F(y_1-x)D_F(x-x)D_F(x-y_2)J(y_1)J(y_2) \\
&+\frac{1}{8}\lambda\int d^4x d^4y_1\, d^4y_2\, d^4y_3\, d^4y_4\, D_F(y_1-x)D_F(y_2-x)D_F(y_3-x)D_F(y_4-x)J(y_1)J(y_2)J(y_3)J(y_4) \\
&+\mathcal{O}(\lambda^2)\,.
\end{aligned}
\tag{2.114}
$$

Eq. (2.113) can be solved for $J(x)$ perturbatively and by performing integration by parts using

$$
(\partial_x^2 + m^2)D_F(x-y) = -i\delta^4(x-y)\,. \tag{2.115}
$$

To calculate $J(x)$ we start off by acting the operator $(\partial_x^2 + m^2)$ on eq. (2.113)

$$
\begin{aligned}
(\partial_x^2 + m^2)\phi_{\text{cl}}(x) &= +i\int d^4y(\partial_x^2 + m^2)D_F(x-y)J(y) \\
&+\frac{1}{2}\lambda\int d^4z\, d^4z'\, (\partial_x^2 + m^2)D_F(x-z)D_F(z-z)D_F(z-z')J(z')
\end{aligned}
$$

$$-\frac{\lambda}{6}\int d^4z d^4z_1' d^4z_2' d^4z_3' (\partial_x^2+m^2)D_F(x-z)D_F(z_1'-z)D_F(z_2'-z)D_F(z_3'-z)J(z_1')J(z_2')J(z_3')$$
$$+\mathcal{O}(\lambda^2)\,, \tag{2.116}$$

which reads

$$(\partial_x^2+m^2)\phi_{\rm cl}(x) = J(x) + \frac{-i\lambda}{2}\int d^4z'\, D_F(0)D_F(x-z')J(z')$$
$$-\frac{-i\lambda}{6}\int d^4z_1' d^4z_2' d^4z_3' D_F(z_1'-x)D_F(z_2'-x)D_F(z_3'-x)J(z_1')J(z_2')J(z_3')$$
$$+\mathcal{O}(\lambda^2)\,. \tag{2.117}$$

This relation can be organized as

$$J(x) = (\partial_x^2+m^2)\phi_{\rm cl}(x) - \frac{-i\lambda}{2}\int d^4z'\, D_F(0)D_F(x-z')J(z')$$
$$+\frac{-i\lambda}{6}\int d^4z_1' d^4z_2' d^4z_3' D_F(z_1'-x)D_F(z_2'-x)D_F(z_3'-x)J(z_1')J(z_2')J(z_3')$$
$$-\mathcal{O}(\lambda^2)\,. \tag{2.118}$$

Now we arrive to the point that it is straightforward to calculate $J(x)$ perturbatively (substituting, for instance, the zero order identification of $J(x)$ in terms of $\phi_{\rm cl}(x)$ when working at order λ). At zero order we have

$$J(x) = (\partial^2+m^2)\phi_{\rm cl}(x) + \mathcal{O}(\lambda)\,, \tag{2.119}$$

and at first order in λ we obtain

$$J(x) = (\partial^2+m^2)\phi_{\rm cl}(x) + \frac{\lambda}{2}D_F(0)\phi_{\rm cl}(x) + \frac{\lambda}{6}[\phi_{\rm cl}(x)]^3 + \mathcal{O}(\lambda^2)\,. \tag{2.120}$$

From (2.120) one can verify that $\phi_{\rm cl}$ satisfies an equation similar to

$$(\partial_x^2+m^2)\phi(x) = -\frac{\lambda}{6}\phi^3(x)\,, \tag{2.121}$$

with the external source added and quantum corrections:

$$(\partial_x^2+m^2)\phi_{\rm cl}(x) = J(x) - \frac{\lambda}{2}D_F(0)\phi_{\rm cl} - \frac{\lambda}{6}\phi_{\rm cl}^3(x)\,. \tag{2.122}$$

If we restore $\hbar$ we would see that the second term in the last equation is proportional to $\hbar$. Substituting (2.120) into (2.114) gives $\Gamma[\phi_{\rm cl}]$ explicitly as a functional of $\phi_{\rm cl}(x)$. Using the following relations:

$$-\frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2)J(y_1)J(y_2)$$
$$= -\frac{i}{2}\int d^4y_1\, d^4y_2 D_F(y_1-y_2)\left((\partial_{y_1}^2+m^2)\phi_{\rm cl}(y_1) + \frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_1) + \frac{\lambda}{6}[\phi_{\rm cl}(y_1)]^3\right)$$
$$\times\left((\partial_{y_2}^2+m^2)\phi_{\rm cl}(y_2) + \frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_2) + \frac{\lambda}{6}[\phi_{\rm cl}(y_2)]^3\right) + \mathcal{O}(\lambda^2)$$
$$= -\frac{1}{2}\int d^4y_1\, \phi_{\rm cl}(y_1)(\partial_{y_1}^2+m^2)\phi_{\rm cl}(y_1)$$

$$-\frac{2}{2}\int d^4y_1\,\phi_{\rm cl}(y_1)\left(+\frac{\lambda}{2}D_F(0)\phi_{\rm cl}(y_1)+\frac{\lambda}{6}[\phi_{\rm cl}(y_1)]^3\right)+\mathcal{O}(\lambda^2)\;, \tag{2.123}$$

$$\begin{aligned}&-\frac{\lambda}{4}\int d^4x d^4y_1\,d^4y_2\,D_F(y_1-x)D_F(x-x)D_F(x-y_2)J(y_1)J(y_2)\\ &=+\frac{\lambda}{4}\int d^4x\,D_F(0)\,(\phi_{\rm cl}(y_1))^2+\mathcal{O}(\lambda^2)\;,\end{aligned} \tag{2.124}$$

$$\begin{aligned}&+\frac{1}{8}\lambda\int d^4xd^4y_1\,d^4y_2\,d^4y_3\,d^4y_4\,D_F(y_1-x)D_F(y_2-x)D_F(y_3-x)D_F(y_4-x)J(y_1)J(y_2)J(y_3)J(y_4)\\ &=+\frac{\lambda}{8}\int d^4x\,D_F(0)\,(\phi_{\rm cl}(y_1))^4+\mathcal{O}(\lambda^2)\;,\end{aligned} \tag{2.125}$$

one can show that

$$\begin{aligned}\Gamma[\phi_{\rm cl}]&=+i\ln N-\frac{\lambda}{8}\int d^4z(D_F(0))^2-\frac{1}{2}\int d^4x\,\phi_{\rm cl}(x)(\partial_x^2+m^2)\phi_{\rm cl}(x)\\ &-\frac{\lambda}{4}D_F(0)\int d^4x(\phi_{\rm cl}(x))^2-\frac{\lambda}{4!}\int d^4x(\phi_{\rm cl}(x))^4+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{2.126}$$

From this equation and from

$$\Gamma^{(n)}(x_1,\,...\,,x_n)=\left.\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\,...\,\delta\phi(x_n)}\right|_{\phi=0}\;, \tag{2.127}$$

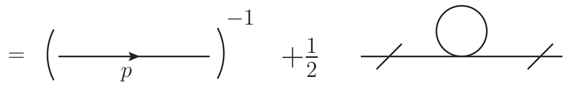
we obtain that

$$\Gamma^{(2)}(x_1,x_2)\;=\;-\left(\partial_{x_1}^2+m^2+\frac{\lambda}{2}D_F(0)\right)\delta(x_1-x_2)\,, \tag{2.128}$$

$$\Gamma^{(4)}(x_1,x_2,x_3,x_4)\;=\;-\lambda\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)\,, \tag{2.129}$$

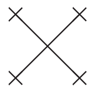
we obtain in momentum space

$$i\Gamma^{(2)}(p,-p)=i(p^2-m^2)-i\frac{\lambda}{2}D_F(0)+\mathcal{O}(\lambda^2)$$



(2.130)

$$i\Gamma^{(4)}(p_1,p_2,p_3,p_4)=-i\lambda$$



. (2.131)

Relation between $G_{\rm conn}^{(n)}$ and $\Gamma_{\rm conn}^{(n)}$

We recall that (we set $\hbar=1$)

$$\Gamma[\phi_{\rm cl}]=W[J]-\int d^4xJ(x)\phi_{\rm cl}(x)\,, \tag{2.132}$$

and we have got

$$\frac{\delta W[J]}{\delta J(x)} = \phi_{\rm cl}(x)\,, \quad \frac{\delta \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)} = -J(x)\,. \tag{2.133}$$

Therefore we get for

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta \phi_{\rm cl}(x)}{\delta J(y)}\,, \quad G^{(2)}_{\rm conn}(x,y) = \frac{1}{i}\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0}\,, \tag{2.134}$$

$$\frac{\delta^2 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)\phi_{\rm cl}(y)} = -\frac{\delta J(x)}{\delta \phi_{\rm cl}(y)}\,, \quad \Gamma^{(2)}(x,y) = \frac{\delta^2 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)\phi_{\rm cl}(y)}\bigg|_{J=0}\,. \tag{2.135}$$

These are clearly one the inverse of the other

$$\begin{aligned}&\int d^4z(-i)\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(z)\phi_{\rm cl}(y)} = i\int d^4z\frac{\delta \phi_{\rm cl}(x)}{\delta J(z)}\frac{\delta J(x)}{\delta \phi_{\rm cl}(y)}\\ &= \frac{\delta \phi_{\rm cl}(x)}{\delta \phi_{\rm cl}(y)} = i\delta^4(x-y)\,,\end{aligned} \tag{2.136}$$

which is then equivalent to (by using the relations in eqs. (2.135))

$$\int d^4z G^{(2)}_{\rm conn}(x,z)\Gamma^{(2)}(z,y) = i\delta^4(x-y)\,, \tag{2.137}$$

$$\Rightarrow G^{(2)}_{\rm conn}(x,y) = i\left(\Gamma^{(2)}\right)^{-1}(x,y)\,. \tag{2.138}$$

We can obtain many relations of this type simply by differentiating (2.136) and using

$$\frac{\delta}{\delta J(x)} = \int d^4y\frac{\delta \phi_{\rm cl}(y)}{\delta J(x)}\frac{\delta}{\delta \phi_{\rm cl}(y)} = \int d^4y\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\frac{\delta}{\delta \phi_{\rm cl}(y)}\,. \tag{2.139}$$

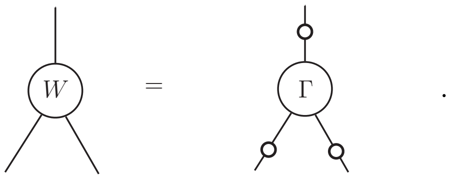
For example let us differentiate eq. (2.136) with respect to $\delta/\delta J(x_1)$:

$$\begin{aligned}&\int d^4z\left(-\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x)\delta J(z)}\right)\frac{\delta^2 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(z)\phi_{\rm cl}(y)}\\ &+\int d^4z\int d^4x'\left(-\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\right)\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x')}\frac{\delta^3 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(z)\delta\phi_{\rm cl}(y)\delta\phi_{\rm cl}(x')} = 0\,.\end{aligned} \tag{2.140}$$

If we now multiply by $(\delta^2 W[J])(\delta J(y)\delta J(x_3))$ and perform the integral over y and use (2.136) we will have (redefining labels)

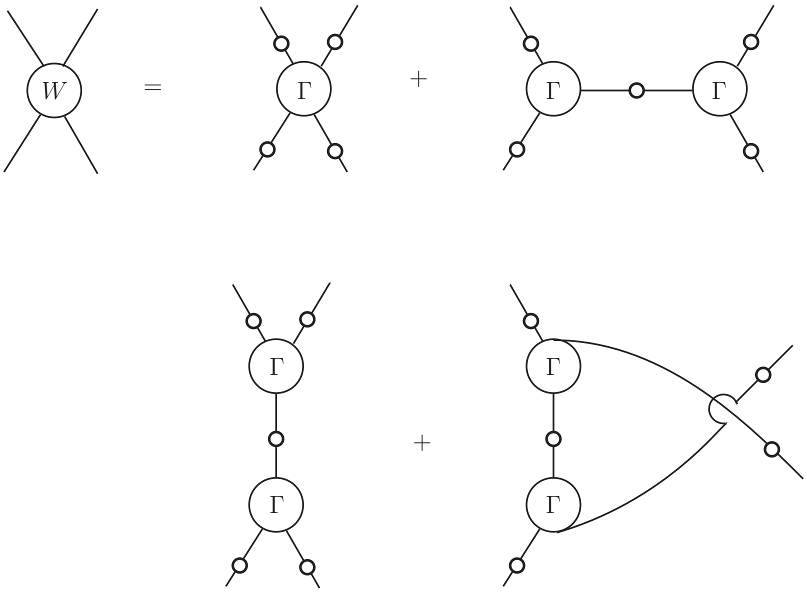
$$\begin{aligned}&\frac{\delta^3 W[J]}{\delta J(y_1)\delta J(y_2)\delta J(y_3)}\\ &= \int d^4x_1\int d^4x_2\int d^4x_3\frac{\delta^2 W[J]}{\delta J(y_1)\delta J(x_1)}\frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)}\frac{\delta^2 W[J]}{\delta J(y_3)\delta J(x_3)}\\ &\times\frac{\delta^3 \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x_1)\delta\phi_{\rm cl}(x_2)\delta\phi_{\rm cl}(x_3)}\,.\end{aligned} \tag{2.141}$$

Eq. (2.141) can be represented as (◯ exact propagators in the external lines)



(2.142)

Differentiating again eq. (2.141) we obtain

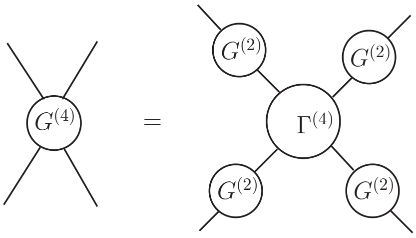


(2.143)

When $J = 0$ several contributions go to zero since, for example, we do not have in $\lambda\phi^4$ theory *1PI diagrams with an odd number of external lines*:

$$\frac{\delta^3\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)\delta\phi_{\rm cl}(y)\delta\phi_{\rm cl}(z)} = 0\,. \tag{2.144}$$

Then we have:

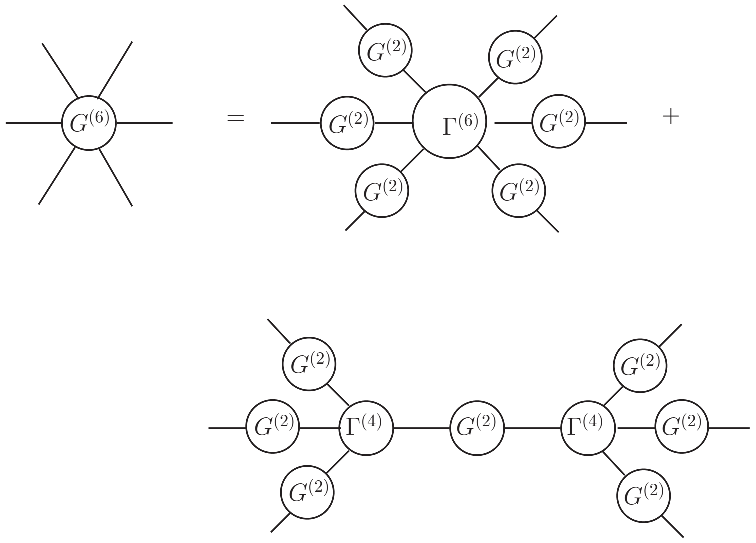


(2.145)

which one writes analytically:

$$\begin{aligned}
&G^{(4)}_{\rm conn}(y_1, y_2, y_3, y_4)\\
&= \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 G^{(2)}_{\rm conn}(x_1 - y_1) G^{(2)}_{\rm conn}(x_2 - y_2) G^{(2)}_{\rm conn}(x_3 - y_3) G^{(2)}_{\rm conn}(x_4 - y_4)\\
&\times i\Gamma^{(4)}(x_1, x_2, x_3, x_4)\,,
\end{aligned} \tag{2.146}$$

and diagrammatically:



(2.147)

These formulas show *the perturbative meaning of* $\Gamma^{(n)}(x_1, \ldots, x_n)$*: 1PI diagrams with amputated external legs.* The only exception is $\Gamma^{(2)}$.

Appendices

A Brief reminder on functional derivatives

A wide range of physics can be formulated in terms of so called variational calculus. The first ingredient is a functional which is a map from a certain space of functions to numbers. Precisely, a functional $F[\phi]$ is a mapping from a normed linear space of functions (a Banach space) $M \equiv \{\phi(x) : x \in \mathbb{R}\}$ to the field of real or complex numbers, $F : M \to \mathbb{R}$ or $\mathbb{C}$.

Example of functionals used in QFT include the action functionals:

$$\begin{aligned} S[q] \quad &= \textstyle\int_{t_{in}}^{t_{fin}} dt L(q, \dot{q}) && \text{(A.1)} \\ S[\phi] \quad &= \textstyle\int dt L(\phi, \partial_\mu \phi) = \int dt d^3x \mathcal{L}(\phi, \partial_\mu \phi) && \text{(A.2)} \end{aligned}$$

where the first line refers to the action of classical mechanics and the second one to the action of field theory. Another example is the class of generating functionals.

The discussion of field theories makes ample use of functional derivatives, i.e. the differentiation of a functional with respect to its argument. Here we give a simple introduction and give some properties of this mathematical operation. If you want to obtain more precise definitions and details I suggest you to look into mathematical books, like for example Gelfand and Fomin, "Calculus of variations" (Dover Books on Mathematics, 2000). or the volume 1 of Courant and Hilbert, "Methods of Mathematical Physics" (Wiley, 1989).

The object $\delta F[\phi]/\delta\phi(x)$ tells how the value of the functional changes if the function $\phi(x)$ is changed at the pont x. We can define the functional differential (or variation) as

$$\delta F[\phi] = \int dx \frac{\delta F[\phi]}{\delta \phi(x)} \delta \phi(x) \tag{A.3}$$

which tells that the total change in F upon variation of the function $\phi(x)$ is a linear superposition of the local changes summed over the whole range of x values. As in ordinary differentiation the functional derivative can be represented as the limit of divided differences. To see this in concrete we construct a specific variation of the independent variable, i.e. the function $\phi(x)$, which is localized at the point y with strength ε:

$$\delta\phi(x) = \varepsilon \delta(x - y). \tag{A.4}$$

Inserting this into $(A.3)$ we have

$$\delta F[\phi] = F[\phi + \varepsilon\delta(x-y)] - F[\phi] = \int dx \frac{\delta F[\phi(x)]}{\delta\phi(x)} \varepsilon\delta(x-y) = \varepsilon \frac{\delta F}{\delta\phi(y)} \tag{A.5}$$

or, in the limit of vanishing ε

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\delta F[\phi(x)]}{\delta\phi(y)} = \lim_{\varepsilon\to 0} \frac{F[\phi(x) + \varepsilon\delta(x-y)] - F[\phi(x)]}{\varepsilon}. \tag{A.6}$$

Most of the rules of ordinary differential calculus can also apply to functional derivatives. We deal with a linear operation. Therefore given two functionals F and G and two constants λ, μ so the functional derivative satisfies

$$\frac{\delta(\lambda F + \mu G)}{\delta\phi(x)} = \lambda\frac{\delta F}{\delta\phi(x)} + \mu\frac{\delta G}{\delta\phi(x)}. \tag{A.7}$$

The derivative of the combined functional $F[\phi] = G[\phi]H[\phi]$ is given by

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\delta G[\phi]}{\delta\phi(x)}H[\phi] + G[\phi]\frac{\delta H[\phi]}{\delta\phi(x)}. \tag{A.8}$$

Similarly the chain rule can be applied to the functional of a functional

$$\frac{\delta}{\delta\phi(y)}F[G[\phi]] = \int dx \frac{\delta F[G]}{\delta G(x)}\frac{\delta G[\phi]}{\delta\phi(y)}. \tag{A.9}$$

Then for a functional $F[\phi]$ given by

$$F[\phi] = \int d^4x_1 \dots d^4x_n f(x_1, \dots, x_n)\phi(x_1)\dots\phi(x_n) \tag{A.10}$$

with f symmetric in all variables, we have:

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \int d^4x_1 \dots d^4x_{n-1}\phi(x_1)\dots\phi(x_{n-1})\, n\, f(x_1, \dots, x_{n-1}, x)\,. \tag{A.11}$$

If the functional is given by the series:

$$F[\phi] = \sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1 \dots d^4x_n f_n(x_1, \dots, x_n)\phi(x_1)\dots\phi(x_n)\,, \tag{A.12}$$

we have

$$f_n(x_1, \dots, x_n) = \left.\frac{\delta^n F[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)}\right|_{J=0}\,. \tag{A.13}$$

B Saddle Point Approximation

Consider the integral

$$I = \int_{-\infty}^{+\infty} dq\; \mathrm{e}^{-f(q)/\hbar}. \tag{B.1}$$

For very small values of $\hbar$ (as $\hbar \to 0$), the integral is dominated by the minimum of $f(q)$ (for simplicity we assume that the integral has only one minimum). Using the power expansion of $f(x)$ about its minimum, denoted by q_0,

$$f(q) = f(q_0) + \frac{1}{2}f''(q_0)\,(q - q_0)^2 + \mathcal{O}(q - q_0)^3\;, \tag{B.2}$$

one can simplify the integral as

$$I \;=\; \mathrm{e}^{-f(q_0)/\hbar}\int_{-\infty}^{+\infty} dq\; \mathrm{e}^{-\frac{1}{2\hbar}f''(q_0)\,(q-q_0)^2 + \mathcal{O}(q-q_0)^3}\;,$$

$$\rightarrow \quad \mathrm{e}^{-f(q_0)/\hbar} \int_{-\infty}^{+\infty} dq \; \mathrm{e}^{-\frac{1}{2\hbar} f''(q_0)\,(q-q_0)^2} \; . \tag{B.3}$$

It is straightforward to calculate the Gaussian integral in the above expression. Using the change of variable $x = (q - q_0)/\sqrt{\hbar}$, one can easily find

$$I = \mathrm{e}^{-f(q_0)/\hbar} \sqrt{\frac{2\pi\hbar}{f''(q_0)}} \; \mathrm{e}^{-\mathcal{O}(\hbar^{1/2})} \; . \tag{B.4}$$

In the case of the action $S\,[x]$, which appears in the path integral calculations, one needs to use the functional derivatives. Similar to the above example, one can see that, as $\hbar \rightarrow 0$, the path integral is dominated by the classical solution

$$\frac{\delta S}{\delta x} = 0 \, , \tag{B.5}$$

with appropriate boundary condition. Therefor, for $\hbar \rightarrow 0$ only the classical solution remains.

C Wick rotation

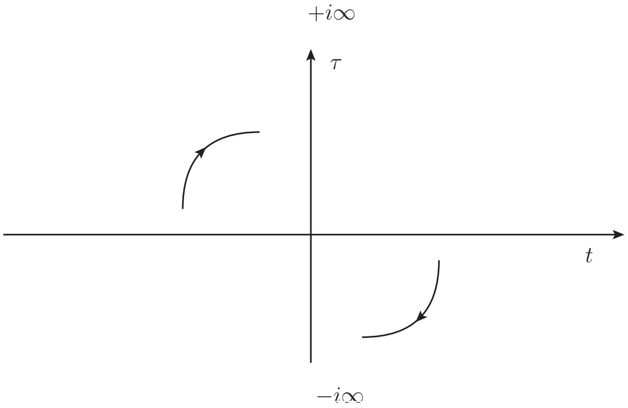
We have seen in the path integral formulation of QFT that might be useful to slightly tip into the complex plane the time integration. This corresponds to an analytic continuation from real to imaginary times. One can think of the rotation for a momentum integral (and then changing the energy of the particle into a complex energy) or for the time coordinate. In the latter case we have

$$t \rightarrow i\tau \, , \tag{C.1}$$

that can be seen also as a rotation of the temporal axis

$$t \rightarrow e^{-i\theta} t = \tau \Rightarrow -it = \tau \Rightarrow t = i\tau \, , \tag{C.2}$$

and for $\theta = \pi/2$ we have the Wick rotation.



D Gaussian integration: analytic continuation

We have defined functional integrals as the appropriate limit of multiple integrals. One typically encounters integrals of the type:

$$I = \int_{-\infty}^{+\infty} dx\, exp(-b\epsilon x^2 + ibx^2) = 2\int_{0}^{+\infty} dx\, exp(-bx^2(\epsilon - i))\,, \tag{D.1}$$

with $\epsilon \to 0$ that corresponds to the $(+i\epsilon)$ prescription.
Rotating the contour of integration $x' = xe^{i\varphi}$ so that $e^{-2i\varphi}(\epsilon - i) = 1$ one gets

$$I = \frac{1}{\sqrt{\epsilon - i}}\left(\frac{\pi}{b}\right)^{\frac{1}{2}} \underset{\epsilon\to 0}{\to} \left(\frac{i\pi}{b}\right)^{\frac{1}{2}}\,, \tag{D.2}$$

which is the analytic continuation of the gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \tag{D.3}$$

for complex $a\,(\mathrm{Re}\, a > 0)$.
A similar result holds for integration over the complex variable $z = x + iy$.

Gaussian integrals

From the known equation:

$$\int_{-\infty}^{+\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}\,, \tag{D.4}$$

we can simply obtain

$$\int_{-\infty}^{+\infty} dx_1 ... dx_n e^{-\frac{1}{2}\sum_{k=1}^{n} a_k x_k^2} = \prod_{k=1}^{n}\left(\frac{2\pi}{a_k}\right)^{\frac{1}{2}}\,. \tag{D.5}$$

We can write this result using scalar products and matrices. We consider $x, y \in \mathbb{R}^n$ so that

$$x\cdot y = (x, y) = \sum_{k=1}^{n} x_k y_k\,, \tag{D.6}$$

and we consider a diagonal matrix

$$A = \begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{pmatrix} \tag{D.7}$$

so that

$$(x, Ax) = \sum_{k=1}^{n} a_k x_k^2\,. \tag{D.8}$$

One can also define the integral measure as

$$[dx] = \frac{dx_1 ... dx_n}{(2\pi)^{\frac{n}{2}}} = \frac{dx^n}{(2\pi)^{\frac{n}{2}}}\,. \tag{D.9}$$

So that we can finally write

$$\int [dx]\, exp\left[-\frac{1}{2}(x, Ax)\right] = \frac{1}{\sqrt{\det A}}\,. \tag{D.10}$$

If A is not a diagonal matrix but it is symmetric, then it can always be diagonalized by an orthogonal transformation:

$$O^{-1}AO = diag(a_1 ... a_n)\,,\ \text{with } O^{-1} = O^T, \quad O \in O(n)\,. \tag{D.11}$$

Then, one can perform on the integral the transformation $x \to Ox$, $dx = dOx = dx$ because the determinant of O is 1 (being an orthogonal matrix). Then we write

$$\begin{aligned}\int [dx]\, exp\left\{-\frac{1}{2}(x, Ax)\right\} &= \int [dOx] exp\left\{-\frac{1}{2}(Ox, AOx)\right\}\\ &= \int [dx]\, exp\left\{-\frac{1}{2}(x, diag(a_1 ... a_n)x)\right\}\\ &= \frac{1}{\sqrt{detA}}\end{aligned} \tag{D.12}$$

where the exponent has been simplified as follows

$$(Ox, AOx) = (x, O^T AOx) = (x, diag(a_1 ... a_n)x)\,. \tag{D.13}$$

Quadratical forms

We can start with the quadratic form

$$q(x) = -\frac{a}{2}x^2 + bx + c = q(x_0) - \frac{a}{2}(x - x_0)^2\,, \tag{D.14}$$

with $x_0 = b/a$ and

$$\begin{cases} \frac{\partial q}{\partial x} = -ax + b, \quad \frac{\partial q}{\partial x}\Big|_{x=x_0} = 0\,,\\ q(x_0) = -a\frac{b^2}{2a^2} + \frac{b^2}{a} + c = \frac{b^2}{2a} + c\,. \end{cases} \tag{D.15}$$

Then we have

$$\int_{-\infty}^{+\infty} dx\, exp(-\frac{a}{2}x^2 + bx + c) = exp(q(x_0))\sqrt{\frac{2\pi}{a}} = exp\left(\frac{b^2}{2a} + c\right)\sqrt{\frac{2\pi}{a}}\,, \tag{D.16}$$

and we can generalize this formula to vectors $x \in \mathbb{R}^n$ and matrices $n \times n$. Then, we have

$$Q(x) = -\frac{1}{2}(x, Ax) + (b, x) + c \tag{D.17}$$

with A positive and invertible matrix. We put

$$x_0 = A^{-1}b \tag{D.18}$$

and so

$$\begin{aligned} Q(x) &= Q(x_0) - \frac{1}{2}(x - x_0, A(x - x_0))\\ &= \frac{1}{2}(b, A^{-1}b) + c - \frac{1}{2}(x - x_0, A(x - x_0)) \end{aligned} \tag{D.19}$$

and we obtain, using (D.16)

$$\int [dx]\, e^{+Q(x)} = exp\left[\frac{1}{2}(b, A^{-1}b) + c\right]\frac{1}{\sqrt{detA}}\,. \tag{D.20}$$

Complex variables and Hermitian matrices

From what we have seen we can calculate

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, e^{-a\frac{(x^2+y^2)}{2}} = \frac{2\pi}{a} \,, \tag{D.21}$$

and we can write $z = \frac{1}{\sqrt{2}}(x+iy)$, $z^* = \frac{1}{\sqrt{2}}(x-iy)$ with $z \in \mathbb{C}$; and $x, y \in \mathbb{R}$. Performing then the change of variables:

$$dxdy = \left|\frac{\partial(x,y)}{\partial(z,z^*)}\right| dzdz^* = \left|\det\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\right| = dz^*dz \,, \tag{D.22}$$

from (D.21) we can obtain

$$\int \frac{dzdz^*}{2\pi} e^{-az^*z} = \frac{1}{a} \tag{D.23}$$

and this can be generalized to

$$\int \frac{dz_1 dz_1^* ... dz_n dz_n^*}{(2\pi)^N} e^{-(z^*, Az)} = (\mathrm{detA})^{-1} = e^{-\mathrm{tr}\ln A} \tag{D.24}$$

for any positive definite matrix A that can be diagonalized by unitary transformation. In the last equality we use a relation which is shown below (see Eq. D.31). To define Gaussian integration over fields we will generalize the results that hold for N-dimensional vector spaces to infinite dimensional functional spaces. For real functions $\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ we define the standard scalar product

$$\begin{aligned} (\phi, \psi) &= \int d^4x \phi(x)\psi(x) \,, \\ A\phi(x) &= \int d^4y A(x,y)\phi(y) \end{aligned} \tag{D.25}$$

Any action quadratic in the fields can be written in general as

$$S = \int d^4x d^4y \phi(x) A(x,y) \phi(y) \tag{D.26}$$

For this set of actions we obtain

$$\int \mathcal{D}\phi \, \exp\left\{-\frac{1}{2}\int d^4x \int d^4y \phi(x)A(x,y)\phi(y)\right\} = \frac{1}{\sqrt{\det A}} \,, \tag{D.27}$$

$$\begin{aligned} &\int \mathcal{D}\phi \, \exp\left\{-\frac{1}{2}\int d^4x \int d^4y \phi(x)A(x,y)\phi(y) + J(x)\phi(x)\right\} \\ &= \frac{1}{\sqrt{\det A}} \exp\left\{\frac{1}{2}\int d^4x \int d^4y J(x)A^{-1}(x,y)J(y)\right\} \,, \end{aligned} \tag{D.28}$$

$$\int \mathcal{D}\phi \int \mathcal{D}\phi^* \, \exp\left\{-\int d^4x \int d^4y \phi^*(x)A(x,y)\phi(y)\right\} = (\det A)^{-1} \,, \tag{D.29}$$

where the 2π factors have been included in the definition of the functional measure (this is not relevant as we will always deal with ratios of such integrals). For any matrix A that can be diagonalized by unitary transformations there are some useful formulas:

$$\begin{cases} \det(1-A) = \exp\left\{\mathrm{tr}\ \ln(1-A)\right\} \,, \\ \det A = \exp\left\{\mathrm{tr}\ \ln A\right\} \,, \\ \mathrm{tr}\ \ln(1-A) = -\mathrm{tr}[A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + ...] \,. \end{cases} \tag{D.30}$$

with

$$\det A = \Pi_i a_i = \exp\left(\ln \Pi_i a_i\right) = \exp\left(\sum_i \ln a_i\right) = \exp\left(\mathrm{tr} \ln A\right) \ , \tag{D.31}$$

where we have used the fact that the determinant of the matrix A is the product of its eigenvalues a_i.