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1 Path Integral Quantization

1.1 Non-relativistic Quantum Mechanics

Let us start with a conceptual discussion, namely the *double slit* experiment (see Fig. 1.1). A particle emitted from a source S at time $t = 0$ passes through one or the other of the two holes A_1 and A_2 drilled in a screen, and it is detected at time $t = T$ by a detector located at O .

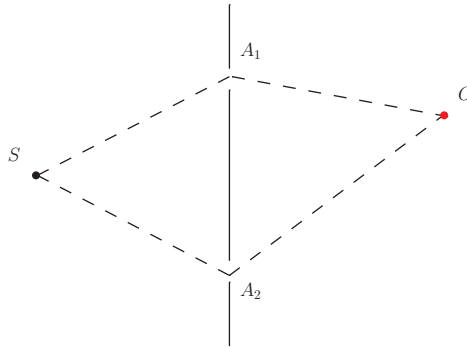


Figure 1.1: Double slit experiment.

According to the postulates of Quantum Mechanics (QM) we may say:

- The probability for the detection in O is given by

$$P(S \rightarrow O) = |A(S \rightarrow O)|^2, \quad (1.1)$$

where A is the amplitude of the process.

- From the superposition principle the amplitude for the detection in O is the sum of the amplitude for the particle to propagate from the source S through the hole A_1 and then onward to the point O , and the amplitude for the particle to propagate from S through A_2 and then to O :

$$A(S \rightarrow O) = A(S \rightarrow A_1 \rightarrow O) + A(S \rightarrow A_2 \rightarrow O). \quad (1.2)$$

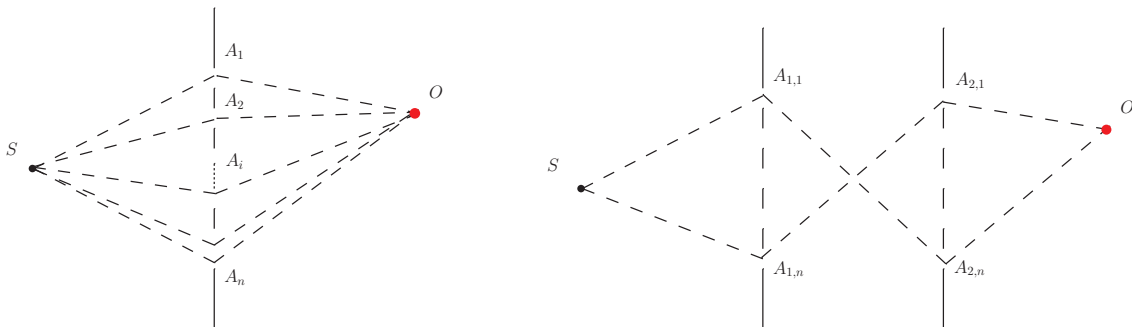


Figure 1.2: Experiment with one screen with n holes (left), experiment with two screens with n holes each (right).

We note that the corresponding probability contains the interference between the two terms in Eq. (1.2). If we drill n holes (see Fig. 1.2) we find for the amplitude the generalization of Eq. (1.2), that reads

$$A(S \rightarrow O) = \sum_{i=1}^n A(S \rightarrow A_i \rightarrow O). \quad (1.3)$$

If we add a second screen with n holes (see again Fig. 1.2) we have for the amplitude

$$A(S \rightarrow O) = \sum_{i,j=1}^n A(S \rightarrow A_{1,i} \rightarrow A_{2,j} \rightarrow O). \quad (1.4)$$

If one would add an infinite numbers of screens, drill infinite holes in each, we will get

$$A(S \rightarrow O) = \sum_{\text{all paths}} A(S \rightarrow \text{path} \rightarrow O), \quad (1.5)$$

that is understood as a sum over all paths (see Fig. 1.3). Hence in order to calculate the amplitude we have to sum over the amplitude for the particle to propagate from the source to the detector following *all* possible paths.

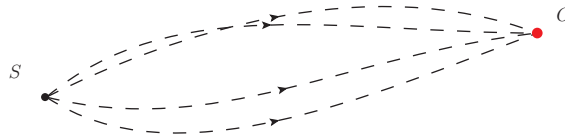


Figure 1.3: All possible paths in the limit of infinite holes and screens.

- 1) *How could one define the sum over the paths?*

We have to consider the functional integral that is a sum over an infinite number of possible trajectories in order to compute a quantum amplitude. We can take a path and approximate it by straight line segments and let the segments go to zero.



- 2) *How can we construct the amplitude along a particular path?*

If we know the amplitude for each infinitesimal segment, then we can just multiply them together to get the amplitude of the whole path.

In QM the amplitude to propagate from a point q_I to a point q_F in a time T is governed by the unitary operator $e^{-i\hat{H}T}$, where \hat{H} is the Hamiltonian. The corresponding amplitude is given by $\langle q_F | e^{-i\hat{H}T} | q_I \rangle$.

1.2 Transition amplitude as path integrals

Feynman, following the ideas from Dirac, has shown that QM could be formulated in terms of path integrals. We discuss this approach to QM in details since it provides the key to the path integral formulation of QFT.

We consider a quantum mechanical system with one degree of freedom: one generalized coordinate, x , and its conjugate momentum, p . We keep $\hbar \neq 1$ and $c = 1$. In the canonical quantization we work with the Hilbert space operators \hat{x} and \hat{p} defined by the commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (1.6)$$

where we also display the Hamiltonian of the system under consideration. The eigenstates of the position operator are introduced as:

$$\begin{cases} \hat{x}_H(t)|x, t\rangle_H = x|x, t\rangle_H, & \text{Heisenberg picture} \\ \hat{x}_S|x\rangle_S = x|x\rangle_S, & \text{Schrödinger picture} \end{cases} \quad (1.7)$$

with the relation between the two pictures that reads

$$\begin{cases} |x, t\rangle_H = e^{\frac{i\hat{H}t}{\hbar}}|x\rangle_S \\ |x\rangle_S = e^{-\frac{i\hat{H}t}{\hbar}}|x, t\rangle_H. \end{cases} \quad (1.8)$$

Since $\hat{x}_H(t)$ is time dependent, so are the eigenstates

$$\hat{x}_H(t) = e^{\frac{i\hat{H}t}{\hbar}} \hat{x}_S e^{-\frac{i\hat{H}t}{\hbar}}. \quad (1.9)$$

The eigenstates are normalized as follows:

$$\begin{cases} \langle x''|x'\rangle = \delta(x'' - x'), & \int_{-\infty}^{+\infty} dx'|x'\rangle\langle x'| = 1, \\ \langle p''|p'\rangle = \delta(p'' - p'), & \int_{-\infty}^{+\infty} dp'|p'\rangle\langle p'| = 1, \\ \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p'x'}{\hbar}}. \end{cases} \quad (1.10)$$

The matrix element ${}_H\langle x', t'|x, t\rangle_H = {}_S\langle x'|e^{-i\frac{\hat{H}(t'-t)}{\hbar}}|x\rangle_S$ corresponds to the transition from the eigenstate $|x, t\rangle$ at time t to the eigenstate $|x', t'\rangle$ at time t' , hence it gives the *transition amplitude*.

This matrix element can be first expressed as a *multiple integral* which will then be used to define the *functional integral* (path integral) via a limiting procedure.

First we divide the time interval $(t' - t)$, characterizing the two states, into $(n + 1)$ parts of equal lengths, ϵ , as follows

$$\begin{cases} (n + 1)\epsilon = (t' - t) \\ t' = (n + 1)\epsilon + t \\ t_j = j\epsilon + t, \quad j = 1, \dots, n. \end{cases} \quad (1.11)$$

and $x_0 = x$ for $t_0 = t$, and $x_{n+1} = x'$ for $t_{n+1} = t'$ are kept fixed. Then we can use the completeness relation at each of the time t_j :

$$\int dx_j |x_j, t_j\rangle_{HH}\langle x_j, t_j| = 1, \quad (1.12)$$

to derive

$${}_H\langle x', t' | x, t \rangle_H = \int dx_1 \cdots dx_n {}_H\langle x', t' | x_n, t_n \rangle_H {}_H\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle_H \cdots {}_H\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle_H \cdots {}_H\langle x_1, t_1 | x, t \rangle_H \quad (1.13)$$

Now we proceed using the following relation

$${}_H\langle x_j, t_j | x_{j-1}, t_{j-1} \rangle_H = \langle x_j | e^{-i\frac{\epsilon \hat{H}}{\hbar}} | x_{j-1} \rangle = \langle x_j | x_{j-1} \rangle - i\frac{\epsilon}{\hbar} \langle x_j | \hat{H} | x_{j-1} \rangle + \mathcal{O}(\epsilon^2), \quad (1.14)$$

where

$$\begin{cases} x_0 = x, t_0 = t \\ x_{n+1} = x', t_{n+1} = t'. \end{cases} \quad (1.15)$$

are fixed. We choose a general form for the Hamiltonian (separable)

$$\hat{H}(x, p) = \hat{f}(p) + \hat{g}(x), \quad (1.16)$$

so that we can write

$$\langle x_j | \hat{H} | x_{j-1} \rangle = \int dp_j \langle x_j | p_j \rangle \langle p_j | \hat{H} | x_{j-1} \rangle = \int \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar} p_j (x_j - x_{j-1})\right] H(p_j, x_{j-1}), \quad (1.17)$$

and now $H(p_j, x_{j-1})$ is a classical number and not an operator any more. Inserting Eq. (1.17) inside Eq. (1.14) we obtain

$$\begin{aligned} {}_H\langle x_j, t_j | x_{j-1}, t_{j-1} \rangle_H &= \int \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar} p_j (x_j - x_{j-1})\right] \left[1 - \frac{i}{\hbar} \epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2) \\ &= \int \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar} p_j (x_j - x_{j-1}) - \frac{i}{\hbar} \epsilon H(p_j, x_{j-1})\right] + \mathcal{O}(\epsilon^2), \end{aligned} \quad (1.18)$$

and composing the infinitesimal transition amplitudes in (1.13) using eqs. (1.14) and (1.18) we find

$${}_H\langle x', t' | x, t \rangle_H = \lim_{n \rightarrow \infty} \int \prod_{j=1}^n dx_j \int \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} \exp\left\{\frac{i}{\hbar} \sum_{j=1}^{n+1} [p_j (x_j - x_{j-1}) - H(p_j, x_{j-1})(t_j - t_{j-1})]\right\} \quad (1.19)$$

where the limit $n \rightarrow \infty$ ($\epsilon \rightarrow 0$) has been taken and terms of order $\mathcal{O}(\epsilon^2)$ neglected. So we obtain the transition amplitude as a path integral

$${}_H\langle x', t' | x, t \rangle_H = \int \mathcal{D}x \mathcal{D}p \exp\left\{\frac{i}{\hbar} \int_t^{t'} d\tau [p\dot{x} - H(p, x)]\right\}, \quad (1.20)$$

with $x(t) = x$ and $x(t') = x'$ fixed. The right-hand side in (1.20) is called functional integral over the phase space

$$\mathcal{D}x = \prod_{j=1}^n dx_j, \quad \mathcal{D}p = \prod_{j=1}^{n+1} \frac{dp_j}{2\pi\hbar} \quad (1.21)$$

in the limit $n \rightarrow \infty$.

If the Hamiltonian is of the simple form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \quad (1.22)$$

it is convenient to perform the p integration in (1.18). We define $\Delta x_j = x_j - x_{j-1}$ and we obtain

$$\begin{aligned} \int \frac{dp_j}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} \left(p_j \Delta x_j - \frac{p_j^2}{2m} \epsilon \right) \right\} &= \frac{1}{2\pi\hbar} \sqrt{\frac{\pi 2m\hbar}{i\epsilon}} \exp \left\{ -\frac{(\Delta x_j)^2 \hbar 2m}{4\hbar^2 \epsilon i} \right\} \\ &= \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi m\hbar}{i\epsilon}} \exp \left\{ \frac{i}{\hbar} \frac{m}{2} \left(\frac{\Delta x_j}{\epsilon} \right)^2 \epsilon \right\} = \frac{1}{C_j} \exp \left\{ \frac{i}{\hbar} \frac{m}{2} \left(\frac{\Delta x_j}{\epsilon} \right)^2 \epsilon \right\}. \end{aligned} \quad (1.23)$$

We used the result for the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + bx} = \sqrt{\frac{\pi}{\alpha}} e^{\frac{b^2}{4\alpha}}. \quad (1.24)$$

Such result can be equivalently obtained making the shift $p_j \rightarrow p_j + \frac{m\Delta x_j}{\epsilon}$, and the prefactor of the exponential in (1.23) reads

$$\frac{1}{C_j} = \int \frac{dp_j}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} \frac{p_j^2}{2m} \epsilon \right\} = \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi m\hbar}{i\epsilon}} \quad (1.25)$$

which is divergent in the limit $\epsilon \rightarrow 0$. However it is compensated by an analogous factor in the path integral. In this way the final result has the form of a functional integral over configuration space:

$${}_H \langle x', t' | x, t \rangle_H = \lim_{n \rightarrow \infty} \frac{1}{C} \int \prod_{j=1}^n dx_j \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2} \frac{(x_j - x_{j-1})^2}{(t_j - t_{j-1})^2} (t_j - t_{j-1}) - V(x_{j-1})(t_j - t_{j-1}) \right] \right\}, \quad (1.26)$$

which reads

$${}_H \langle x', t' | x, t \rangle_H = \frac{1}{C} \int \mathcal{D}x \exp \left\{ \frac{i}{\hbar} S[x] \right\}, \quad (1.27)$$

where $S[x] = \int_t^{t'} L(x, \dot{x}) d\tau$ is the action integral over the trajectory $x(\tau)$ with the Lagrangian $L(x, \dot{x}) = m\dot{x}^2/2 - V(x)$. The normalization factor is given by

$$\frac{1}{C} = \prod_{j=1}^N \frac{1}{C_j} = \int \mathcal{D}p \exp \left\{ -\frac{i}{\hbar} \int_t^{t'} \frac{p^2}{2m} d\tau \right\}. \quad (1.28)$$

So we finally derived (1.27) *starting from a canonically quantized theory described by the Hamiltonian* (1.22). We can use another approach, define the quantum theory by the functional integral in (1.27), or in other words we can choose the path integral formulation as the quantization prescription for a system with the classical Hamiltonian in the form of (1.22). Then our derivation proves the equivalence of the *path integral* and *canonical quantization* methods for systems described by the Hamiltonian in (1.22).

From now on we will consider quantum theories defined by the path integral formulation. Let us consider

$${}_H \langle x', t' | x, t \rangle_H = \frac{1}{C} \int \mathcal{D}x \exp \left\{ \frac{i}{\hbar} S[x] \right\}, \quad (1.29)$$

if happens that

$$S[x(\tau)] \gg \hbar, \text{ i.e. } \hbar \rightarrow 0 \quad (1.30)$$

we can evaluate (1.29) by using the saddle point approximation. For $\hbar \rightarrow 0$ all the configurations far from the extrema of S give contributions that oscillate wildly under

small deformations of the path and thus give zero contribution. The only first non-vanishing contribution comes from the extrema of the action,

$$\frac{\delta S}{\delta x} = 0, \quad (1.31)$$

which leads to the Euler-Lagrange equation

$$\frac{\partial L(x, \dot{x})}{\partial x} - \frac{d}{dt} \frac{\partial L(x, \dot{x})}{\partial \dot{x}} = 0. \quad (1.32)$$

This is the equation that defines the classical trajectory and the classical equations of motion in classical mechanics.

To obtain the semi-classical solution, one can use the power series expansion of the functional $S[x]$ about its classical solution

$$\begin{aligned} S[x] &= S[x_{\text{cl}}] + S'[x](x - x_{\text{cl}}) + \frac{1}{2}S''[x](x - x_{\text{cl}})^2 + \mathcal{O}(x - x_{\text{cl}})^3 \\ &= S[x_{\text{cl}}] + \frac{1}{2}S''[x](x - x_{\text{cl}})^2 + \mathcal{O}(x - x_{\text{cl}})^3, \end{aligned} \quad (1.33)$$

where

$$\begin{aligned} S'[x] &= \frac{\delta S}{\delta x} \\ S''[x] &= \frac{\delta^2 S}{\delta x^2}. \end{aligned} \quad (1.34)$$

Truncating the above expansion and putting it in Eq. 1.29, one can see that

$$\frac{1}{C} \int \mathcal{D}x \exp \left\{ \frac{i}{\hbar} S[x] \right\} \rightarrow \frac{1}{C} \exp \left\{ \frac{i}{\hbar} S[x_{\text{cl}}] \right\} \int \mathcal{D}x \exp \left\{ \frac{i}{2\hbar} S''[x](x - x_{\text{cl}})^2 \right\}. \quad (1.35)$$

The term $\int \mathcal{D}x \exp \left\{ \frac{i}{2\hbar} S''[x](x - x_{\text{cl}})^2 \right\}$ is the semi-classical contribution to the path integral. As a matter of fact, the path integral formulation allows a relatively simple understanding of the classical and semi-classical limit. As we increase \hbar the classical trajectory still dominates but there are other paths close to it whose action is within $\Delta S \simeq \hbar$ and contribute significantly to the amplitude. The particle does an excursion around the classical trajectory.

The matrix element ${}_H \langle x', t' | T \hat{x}(t) | x, t \rangle_H$ determines all the transition probabilities between quantum mechanical states. In view of the applications of the functional formalism to quantum field theories, it is important to know the path integral representation of the *matrix element of the position operators* that corresponds to the field operators of QFT.

For the time-ordered product of n such operators holds the formula

$${}_H \langle x', t' | T \hat{x}(t_1) \dots \hat{x}(t_n) | x, t \rangle_H = \int \mathcal{D}x \mathcal{D}p \ x(t_1) \dots x(t_n) e^{\frac{i}{\hbar} \int_t^{t'} d\tau [p\dot{x} - H(p, x)]}. \quad (1.36)$$

Here the time-ordered product is defined as

$$T \hat{x}(t_1) \hat{x}(t_2) \dots \hat{x}(t_n) = \hat{x}(\tau_1) \hat{x}(\tau_2) \dots \hat{x}(\tau_n), \quad (1.37)$$

where $\tau_i > \tau_{i+1}$ and $\tau_i = t_j$. We can show that (1.36) is true for the case of two operators $\hat{x}(\tau_1)\hat{x}(\tau_2)$ for $\tau_1 > \tau_2$. We divide again the time interval $(t' - t)$ into small intervals choosing $t_1 \dots t_n$ such that

$$\tau_1 = t_{i_1}, \quad \tau_2 = t_{i_2}, \quad (1.38)$$

and apply the completeness relation at each t_i . We have

$$\begin{aligned} {}_H\langle x', t' | \hat{x}(\tau_1)\hat{x}(\tau_2) | x, t \rangle_H &= \int \prod_i dx_i {}_H\langle x', t' | x_n, t_n \rangle_H \cdots {}_H\langle x_{i_1}, t_{i_1} | \hat{x}(\tau_1) | x_{i_1-1}, t_{i_1-1} \rangle_H \\ &\quad \cdots {}_H\langle x_{i_2}, t_{i_2} | \hat{x}(\tau_2) | x_{i_2-1}, t_{i_2-1} \rangle_H \cdots {}_H\langle x_1, t_1 | x, t \rangle_H \\ &= \int \prod_i dx_i x_{i_1} x_{i_2} {}_H\langle x', t' | x_n, t_n \rangle_H \cdots {}_H\langle x_1, t_1 | x, t \rangle_H. \end{aligned} \quad (1.39)$$

Then if we proceed exactly as before we end up with (1.36). In particular, Eq. (1.39) is true when $\tau_1 > \tau_2$. When $\tau_2 > \tau_1$ the left-hand side of (1.39) is ${}_H\langle x', t' | \hat{x}(\tau_2)\hat{x}(\tau_1) | x, t \rangle_H$. So the path integral is equal to

$$\int \mathcal{D}x \mathcal{D}p \ x(\tau_1)x(\tau_2) e^{\frac{i}{\hbar} \int_t^{t'} d\tau [p\dot{x} - H(p, x)]} = \begin{cases} {}_H\langle x', t' | \hat{x}(\tau_1)\hat{x}(\tau_2) | x, t \rangle_H, & \tau_1 > \tau_2 \\ {}_H\langle x', t' | \hat{x}(\tau_2)\hat{x}(\tau_1) | x, t \rangle_H, & \tau_2 > \tau_1. \end{cases} \quad (1.40)$$

As before it is possible to go from phase space path integrals to the path integrals over configuration space.

1.2.1 Transition amplitude in the presence of an external source $J(\tau)$

The transition amplitude with an external source reads

$${}_H\langle x', t' | x, t \rangle_H^J = \int \mathcal{D}x \mathcal{D}p \ exp \left\{ \frac{i}{\hbar} \int_t^{t'} d\tau [p\dot{x} - H(p, x) + \hbar J(\tau)x(\tau)] \right\}, \quad (1.41)$$

where we have modified the Hamiltonian with a source term $H \rightarrow H - \hbar Jx$. This transition amplitude in presence of an external source can be used as *generating functional* of the matrix element of the position operators. They are given by its functional derivatives with respect to $J(\tau)$:

$${}_H\langle x', t' | T \hat{x}(t_1) \dots \hat{x}(t_n) | x, t \rangle_H = \left(\frac{1}{i} \right)^n \frac{\delta^n}{\delta J(t_1) \cdots \delta J(t_n)} {}_H\langle x', t' | x, t \rangle_H^J \Big|_{J=0} \quad (1.42)$$

where the $\delta/\delta J(t)$ are the functional derivatives.

1.3 Vacuum to vacuum transitions in imaginary time formalism

In this section we do not explicitly indicate the Heisenberg vectors with the subscript H in order not to make the formulas too cumbersome. However all the bras and kets where both the position and the time appear are understood as bras and kets in the Heisenberg picture. In the case when the time appears in the bra and the ket, those are in the Heisenberg representation even if it is not declared with superscripts.

In QFT we are interested in the calculation of Green's functions that are matrix elements of field operators taken between *vacuum states*. It is then useful to consider the analogous problem in QM.

We assume that the Lagrangian of the system is time independent. The energy eigenstates are defined as $\hat{H}|n\rangle = E_n|n\rangle$. In configuration space (Schrödinger description) they correspond to the wave functions $\phi_n(x) = \langle x|n\rangle$. The ground state or vacuum state is described by the wave function $\phi_0(x) = \langle x|0\rangle$.

We have

$$\phi_0(t, x) = e^{-\frac{i}{\hbar}E_0t}{}_S\langle x|0\rangle = \langle x|e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle = {}_H\langle x, t|0\rangle. \quad (1.43)$$

We are interested in $\langle 0|T\hat{x}(t_1) \dots \hat{x}(t_n)|0\rangle$. It can be written, by inserting two completeness relations $\int dx' |x', t'\rangle\langle x', t'| = 1$ and $\int dx |x, t\rangle\langle x, t| = 1$, as follows

$$\langle 0|T\hat{x}(t_1) \dots \hat{x}(t_n)|0\rangle = \int dx' dx \phi_0^*(t', x')\langle x', t'|T\hat{x}(t_1) \dots \hat{x}(t_n)|x, t\rangle\phi_0(t, x) \quad (1.44)$$

and for the transition amplitude in (1.44) we can use the result given in (1.36). The vacuum expectation value written in (1.44) can also be obtained from a generating functional

$$\langle 0|T\hat{x}(t_1) \dots \hat{x}(t_n)|0\rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1) \dots \delta J(t_n)} Z[J] \Big|_{J=0} \quad (1.45)$$

with

$$Z[J] = \langle 0|0\rangle^J = \int dx' dx \phi_0^*(t', x')\langle x', t'|x, t\rangle^J \phi_0(t, x). \quad (1.46)$$

One may understand the definition of the generating functional without the external current as follows

$$\langle 0| \int dx' |x', t'\rangle\langle x', t'| \int dx |x, t\rangle\langle x, t|0\rangle = \int dx' dx \phi_0^*(t', x')\phi_0(t, x)\langle x', t'|x, t\rangle = Z[0], \quad (1.47)$$

then one can just replace $\langle x', t'|x, t\rangle$ with $\langle x', t'|x, t\rangle^J$ that is in turn given in (1.41).

It is very important to derive the generating functional $Z[J]$ in a different way. We are going to show that

$$Z[J] = \lim_{\substack{T_1 \rightarrow +i\infty \\ T_2 \rightarrow -i\infty}} \frac{\exp[(i/\hbar)E_0(T_2 - T_1)]}{\phi_0^*(x_1)\phi_0(x_2)} \langle x_2, T_2|x_1, T_1\rangle^J. \quad (1.48)$$

This implies that $Z[J]$ for $T_1 \rightarrow +i\infty$ and $T_2 \rightarrow -i\infty$ is determined by the transition amplitude $\langle x_2, T_2|x_1, T_1\rangle^J$ at any given x_1, x_2 (for example $x_1 = x_2 = 0$, it does not matter which values we choose for x_1 and x_2) provided that the analytic continuation to the imaginary values of T_1 and T_2 is done.

To prove eq. (1.48) we choose a source $J(\tau)$ that vanishes outside the time interval (t, t') with $T_2 > t' > t > T_1$. Then we can write

$$\langle x_2, T_2|x_1, T_1\rangle^J = \int dx' dx \langle x_2, T_2|x', t'\rangle\langle x', t'|x, t\rangle^J \langle x, t|x_1, T_1\rangle \quad (1.49)$$

where all the states are understood in the Heisenberg picture and we write

$${}_H\langle x, t|x_1, T_1\rangle_H = \langle x|e^{-\frac{i}{\hbar}H(t-T_1)}|x_1\rangle =$$

$$\begin{aligned}
&= \sum_n \langle x | e^{-\frac{i}{\hbar} H(t-T_1)} | n \rangle \langle n | x_1 \rangle \\
&= \sum_n \sum_{n'} \langle x | n' \rangle \langle n' | e^{-\frac{i}{\hbar} H(t-T_1)} | n \rangle \langle n | x_1 \rangle = \\
&= \sum_n \langle x | n \rangle \langle n | x_1 \rangle e^{-\frac{i}{\hbar} E_n(t-T_1)} \\
&= \sum_n \phi_n(x) \phi_n^*(x_1) e^{-\frac{i}{\hbar} E_n(t-T_1)} \longrightarrow \phi_0(x) \phi_0^*(x_1) e^{-\frac{i}{\hbar} E_0(t-T_1)} \quad (1.50)
\end{aligned}$$

where we have inserted two complete set of eigenstates to obtain the result. Similarly we can do the same for ${}_H \langle x_2, T_2 | x', t' \rangle_H$. The only T -dependent terms is now the factor $e^{-\frac{i}{\hbar} E_n(t-T_1)}$ and we can continue $T \rightarrow +i\infty$ explicitly.

The oscillatory behaviour of the exponential is not well defined and we need to make some more precise statement in order to evaluate it unambiguously. To define it one can continue the time to imaginary values and then continue back to the real time at the end ($t = i\tau$).

In the limit $T \rightarrow \infty$ in the sum $\sum_n e^{-E_n T}$ only E_0 which is the lowest energy level survives. For this reason we can write, by using the result in (1.50) :

$$\lim_{T_1 \rightarrow +i\infty} e^{-\frac{i}{\hbar} E_0 T_1} \langle x, t | x_1, T_1 \rangle = \phi_0(x) e^{-\frac{i}{\hbar} E_0 t} \phi_0^*(x_1) = \phi_0(t, x) \phi_0^*(x_1). \quad (1.51)$$

In the same way we calculate

$$\lim_{T_2 \rightarrow -i\infty} e^{\frac{i}{\hbar} E_0 T_2} \langle x_2, T_2 | x', t' \rangle = \phi_0^*(t', x') \phi_0(x_2). \quad (1.52)$$

Then using (1.51) and (1.52) and the definition (1.46) in (1.49) we obtain

$$\begin{aligned}
\langle x_2, T_2 | x_1, T_1 \rangle^J &= \int dx' dx \langle x_2, T_2 | x', t' \rangle \langle x', t' | x, t \rangle^J \langle x, t | x_1, T_1 \rangle \\
&= \int dx' dx e^{-\frac{i}{\hbar} E_0 T_2} \phi_0^*(t', x') \phi_0(x_2) \langle x', t' | x, t \rangle^J e^{\frac{i}{\hbar} E_0 T_1} \phi_0(x, t) \phi_0^*(x_1) \\
&= e^{-\frac{i}{\hbar} E_0 (T_2 - T_1)} \phi_0(x_2) \phi_0^*(x_1) Z[J]. \quad (1.53)
\end{aligned}$$

We have therefore obtained (1.48) back and we have proven that $Z[J]$ can be given also in terms of (1.48).

Then if (1.48) is true we will also have

$$\langle 0 | T x(t_1) \dots x(t_n) | 0 \rangle = \lim_{\substack{T_1 \rightarrow +i\infty \\ T_2 \rightarrow -i\infty}} \frac{e^{\frac{i}{\hbar} E_0 (T_2 - T_1)}}{\phi_0^*(x_1) \phi_0(x_2)} \langle x_2, T_2 | T x(t_1) \dots x(t_n) | x_1, T_1 \rangle. \quad (1.54)$$

Then the vacuum matrix elements can be calculated by taking functional derivatives of the generating functional $Z[J]$ given by (1.48). The J -independent factors are irrelevant because we can always consider quantities like

$$\frac{1}{\langle 0 | 0 \rangle} \langle 0 | T x(t_1) \dots x(t_n) | 0 \rangle = \left(\frac{1}{i} \right)^n \frac{1}{Z[0]} \frac{\delta^n}{\delta J(t_1) \dots \delta J(t_n)} Z[J] \Big|_{J=0}. \quad (1.55)$$

Then we can simply write in place of (1.48):

$$Z[J] = \text{const} \lim_{\substack{T_1 \rightarrow +i\infty \\ T_2 \rightarrow -i\infty}} \int \mathcal{D}x \exp \left\{ \frac{i}{\hbar} \int_{T_1}^{T_2} [L(x, \dot{x}) + \hbar J x] dt \right\}, \quad (1.56)$$

where x_1 and x_2 are arbitrary. The path integral is over all $x(t)$ satisfying the boundary conditions $\lim_{T_1 \rightarrow i\infty} x(T_1) = x_1$ and $\lim_{T_2 \rightarrow -i\infty} x(T_2) = x_2$. Moreover x_1 and x_2 are any chosen constants but often are taken to be zero.

1.4 Path integral formulation of Quantum Field Theory

Green's functions as path integrals

The results that we have obtained up to now are simple to generalize to the case of more than one degree of freedom. If *the number of degrees of freedom is to be d* , the coordinate x should be replaced by a d -component vector. The functional integral now corresponds to the sum over all trajectories in the n -dimensional configuration space, satisfying appropriate boundary conditions. For example, for $p \rightarrow \mathbf{p}$ and $x \rightarrow \mathbf{x}$ in $d = 3$ we have

$$\prod_{k=1}^{N+1} dp_k \rightarrow \prod_{k=1}^{N+1} dp_k^1 \prod_{k=1}^{N+1} dp_k^2 \prod_{k=1}^{N+1} dp_k^3, \quad (1.57)$$

$$\prod_{h=1}^N dx_h \rightarrow \prod_{h=1}^N dx_h^1 \prod_{h=1}^N dx_h^2 \prod_{h=1}^N dx_h^3. \quad (1.58)$$

In Field Theory, the trajectory $x(t)$ is replaced by a field function $\phi(t, \mathbf{x})$. The degrees of freedom are labelled by a continuous index \mathbf{x} , the number of degrees of freedom is *infinite*. In this case to define the appropriate path integral one can start from a *multiple* integral on a *discrete*, and for the moment *finite*, lattice of space-time points.

Here we need to insert the definition of the discretized generating functional. To this aim one divides the space into 4-dimensional cubes of volume ε^4 and identify each degree of freedom with discrete labels, that in turn enters the field and its derivative as follows

$$\phi_n \simeq \phi(x_i, y_j, z_k, t_l), \quad (1.59)$$

$$\left. \frac{\partial \phi}{\partial x} \right|_{i,j,k,l} \simeq \frac{1}{\varepsilon} [\phi(x_i + \varepsilon, y_j, z_k, t_l) - \phi(x_i, y_j, z_k, t_l)]. \quad (1.60)$$

If then put $n = i, j, k, l$ we can write

$$\mathcal{L}(\phi(x_i, y_j, z_k, t_l), \partial_\mu \phi(x_i, y_j, z_k, t_l)) = \mathcal{L}(\phi_n, \partial_\mu \phi_n) = \mathcal{L}_n, \quad (1.61)$$

and also for the generating functional

$$Z[J] = \lim_{N \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int \prod_n^{N^4} d\phi_n \exp \left\{ i \sum_{n=1}^{N^4} \varepsilon^4 (\mathcal{L}_n + \hbar J_n \phi_n) \right\}. \quad (1.62)$$

This corresponds to defining QFT as a limit of a theory with only a finite number of degrees of freedom. The limit of an infinite lattice, which is related to the thermodynamical limit of statistical mechanics, already defines a theory with an infinite number of degrees of freedom.

However this lattice theory does not have the usual Lorentz space-time invariance (it has a different symmetry in space-time) and one has to send the lattice step to zero and the

coupling to a continuous theory. This continuous limit is accompanied by infinities, the *ultraviolet* (UV) divergences of QFT (UV divergences \rightarrow related to high energies/small distances). Then the definition of the functional integral in QFT is more ambiguous than in the case of quantum mechanics.

QFT is formulated in terms of vacuum expectation values of time-ordered products of field operators, the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle. \quad (1.63)$$

We will write down the path integral representation for the $G^{(n)}$ using what we have developed up to here in QM. It is particularly important to remember the role played by Eqs (1.48) and (1.54) in getting rid of the vacuum wave functions that are originally present in (1.44) as boundary conditions.

By analogy with the previous result we can postulate this path integral representation:

$$G^{(n)}(x_1, \dots, x_n) \sim \int \mathcal{D}\phi \phi(x_1)\dots\phi(x_n) e^{\frac{i}{\hbar} \int d^4x \mathcal{L}}, \quad (1.64)$$

where $\mathcal{D}\phi$ denotes the integration over all the functions $\phi(t, \mathbf{x})$ of space and time because for each value of \mathbf{x} , $\phi(t, \mathbf{x})$ corresponds to a separate degree of freedom. Moreover the path integral is understood over all functions that satisfy the boundary conditions

$$\begin{cases} \phi(T_1, \mathbf{x}) = \phi_1(x), \\ \phi(T_2, \mathbf{x}) = \phi_2(x), \end{cases} \quad (1.65)$$

for $T_1 \rightarrow +i\infty$ and $T_2 \rightarrow -i\infty$, and $\phi_1(x)$ and $\phi_2(x)$ are arbitrary and can be put to zero. In fact (1.54) suggests that the boundary conditions are irrelevant.

In Eq. (1.64) we have an oscillatory behaviour. To make well defined calculations we can proceed in two ways:

- 1) Wick rotation: analytic continuation to imaginary time region, namely

$$\tau = -it, \quad t = i\tau. \quad (1.66)$$

In this way one obtains a particularly convenient path integral representation because the weight factor in the integral, $\exp(-S_E/\hbar)$, is non-negative. Indeed we have that

$$S_E[\phi] = \int d^4x_E \mathcal{L}(x_E) = -i(-1) \int d^4x (-\mathcal{L}(x)) = -iS[\phi] \quad (1.67)$$

taking into account that $d^4x_E = -id^4x$, that the change in time integral limits give a minus and that

$$\mathcal{L}(x_E) = \frac{1}{2} [(\partial_\mu^E \phi)^2 + m^2 \phi^2] = -\frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) = -\mathcal{L}. \quad (1.68)$$

Then the Euclidean path integral formalism can be used to define Minkowski space Green's functions by analytic continuation of the Euclidean ones. So (1.64) may be understood as an analytic continuation in the variables t_1, \dots, t_n of the analogous Euclidean formula.

- 2) One can work in Minkowski space and regularize the integrand in the path integral by adding a small imaginary piece to the Lagrangian density \mathcal{L} ($+i\epsilon$ prescription)

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) + \frac{1}{2} i\epsilon \phi^2 \quad (1.69)$$

Eq. (1.64) has to be regarded as the formulation of the theory.

Now, what is the relation between this path integral definition and the usual canonical operator formulation of the QFT based on the same \mathcal{L} ?

For our present scopes: they are equivalent if they originate the same perturbation theory and thus the same Feynman rules. We will check this in the following. The derivation of perturbation theory is much simpler in the functional framework, especially in the case of gauge field theories.

It is convenient to normalize the Green's functions by factorizing out the vacuum amplitude:

$$G^{(n)}(x_1, \dots, x_n) = \frac{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle}{\langle 0|0\rangle} = N \int \mathcal{D}\phi \phi(x_1)\dots\phi(x_n) \exp \left\{ \frac{i}{\hbar} \int d^4x \mathcal{L} \right\}, \quad (1.70)$$

where

$$\frac{1}{N} = \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \int d^4x \mathcal{L} \right\} \sim \langle 0|0\rangle, \quad (1.71)$$

so that extra factors like those appearing in (1.54) are eliminated.

The Green's function in (1.70) are given *by the functional derivatives of the generating functional $Z[J]$ that gives the vacuum transition in the presence of an external sources:*

$$Z[J] = N \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \int d^4x [\mathcal{L} + \hbar J(x)\phi(x)] \right\}, \quad (1.72)$$

and expanding $Z[J]$ in powers of J around $J = 0$ we obtain

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n G^{(n)}(x_1, \dots, x_n) J(x_1) \dots J(x_n) \quad (1.73)$$

and from this

$$G^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{i} \right)^n \frac{\delta^n}{\delta J(t_1) \dots \delta J(t_n)} Z[J] \Big|_{J=0}. \quad (1.74)$$

The Green's functions can also be considered as the analytic continuation of those obtained from the generating functional defined in Euclidean space with $x_0 = -ix_4$ ($t = i\tau$) with x_4 real:

$$Z_E[J] = N \int \mathcal{D}\phi \exp \left\{ -\frac{1}{\hbar} S_E[\phi(x)] + \int d^4x_E J(x)\phi(x) \right\}, \quad (1.75)$$

where $d^4x_E = dx_1 dx_2 dx_3 dx_4$ and $S_E[\phi(x_E)] = (1/2) \int d^4x_E [((\partial\phi)/(\partial x_4))^2 + (\nabla\phi)^2 + m^2\phi^2]$. For a scalar field we have for example

$$G_E^{(n)}(x_1 \dots x_n) = \frac{\delta^n}{\delta J(t_1) \dots \delta J(t_n)} Z_E[J] \Big|_{J=0}. \quad (1.76)$$

Using the path integral formalism one can obtain the equations of motion (see exercise), for example in free Euclidean scalar field theory.

Appendices

A Brief reminder on functional derivatives

A wide range of physics can be formulated in terms of so called variational calculus. The first ingredient is a functional which is a map from a certain space of functions to numbers. Precisely, a functional $F[\phi]$ is a mapping from a normed linear space of functions (a Banach space) $M \equiv \{\phi(x) : x \in \mathbb{R}\}$ to the field of real or complex numbers, $F : M \rightarrow \mathbb{R}$ or \mathbb{C} .

Example of functionals used in QFT include the action functionals:

$$S[q] = \int_{t_{in}}^{t_{fin}} dt L(q, \dot{q}) \quad (\text{A.1})$$

$$S[\phi] = \int dt L(\phi, \partial_\mu \phi) = \int dt d^3x \mathcal{L}(\phi, \partial_\mu \phi) \quad (\text{A.2})$$

where the first line refers to the action of classical mechanics and the second one to the action of field theory. Another example is the class of generating functionals.

The discussion of field theories makes ample use of functional derivatives, i.e. the differentiation of a functional with respect to its argument. Here we give a simple introduction and give some properties of this mathematical operation. If you want to obtain more precise definitions and details I suggest you to look into mathematical books, like for example Gelfand and Fomin, "Calculus of variations" (Dover Books on Mathematics, 2000). or the volume 1 of Courant and Hilbert, "Methods of Mathematical Physics" (Wiley, 1989).

The object $\delta F[\phi]/\delta\phi(x)$ tells how the value of the functional changes if the function $\phi(x)$ is changed at the point x . We can define the functional differential (or variation) as

$$\delta F[\phi] = \int dx \frac{\delta F[\phi]}{\delta\phi(x)} \delta\phi(x) \quad (\text{A.3})$$

which tells that the total change in F upon variation of the function $\phi(x)$ is a linear superposition of the local changes summed over the whole range of x values. As in ordinary differentiation the functional derivative can be represented as the limit of divided differences. To see this in concrete we construct a specific variation of the independent variable, i.e. the function $\phi(x)$, which is localized at the point y with strength ε :

$$\delta\phi(x) = \varepsilon\delta(x - y). \quad (\text{A.4})$$

Inserting this into (A.3) we have

$$\delta F[\phi] = F[\phi + \varepsilon\delta(x - y)] - F[\phi] = \int dx \frac{\delta F[\phi(x)]}{\delta\phi(x)} \varepsilon\delta(x - y) = \varepsilon \frac{\delta F}{\delta\phi(y)} \quad (\text{A.5})$$

or, in the limit of vanishing ε

$$\frac{\delta F[\phi]}{\delta\phi(y)} = \frac{\delta F[\phi(x)]}{\delta\phi(y)} = \lim_{\varepsilon \rightarrow 0} \frac{F[\phi(x) + \varepsilon\delta(x - y)] - F[\phi(x)]}{\varepsilon}. \quad (\text{A.6})$$

Most of the rules of ordinary differential calculus can also apply to functional derivatives. We deal with a linear operation. Therefore given two functionals F and G and two constants λ, μ so the functional derivative satisfies

$$\frac{\delta(\lambda F + \mu G)}{\delta\phi(x)} = \lambda \frac{\delta F}{\delta\phi(x)} + \mu \frac{\delta G}{\delta\phi(x)}. \quad (\text{A.7})$$

The derivative of the combined functional $F[\phi] = G[\phi]H[\phi]$ is given by

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{\delta G[\phi]}{\delta\phi(x)} H[\phi] + G[\phi] \frac{\delta H[\phi]}{\delta\phi(x)}. \quad (\text{A.8})$$

Similarly the chain rule can be applied to the functional of a functional

$$\frac{\delta}{\delta\phi(y)} F[G[\phi]] = \int dx \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi]}{\delta\phi(y)}. \quad (\text{A.9})$$

Then for a functional $F[\phi]$ given by

$$F[\phi] = \int d^4x_1 \dots d^4x_n f(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) \quad (\text{A.10})$$

with f symmetric in all variables, we have:

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \int d^4x_1 \dots d^4x_{n-1} \phi(x_1) \dots \phi(x_{n-1}) n f(x_1, \dots, x_{n-1}, x). \quad (\text{A.11})$$

If the functional is given by the series:

$$F[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n f_n(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n), \quad (\text{A.12})$$

we have

$$f_n(x_1, \dots, x_n) = \left. \frac{\delta^n F[\phi]}{\delta\phi(x_1) \dots \delta\phi(x_n)} \right|_{J=0}. \quad (\text{A.13})$$

B Saddle Point Approximation

Consider the integral

$$I = \int_{-\infty}^{+\infty} dq e^{-f(q)/\hbar}. \quad (\text{B.1})$$

For very small values of \hbar (as $\hbar \rightarrow 0$), the integral is dominated by the minimum of $f(q)$ (for simplicity we assume that the integral has only one minimum). Using the power expansion of $f(x)$ about its minimum, denoted by q_0 ,

$$f(q) = f(q_0) + \frac{1}{2} f''(q_0) (q - q_0)^2 + \mathcal{O}(q - q_0)^3, \quad (\text{B.2})$$

one can simplify the integral as

$$I = e^{-f(q_0)/\hbar} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2\hbar} f''(q_0) (q - q_0)^2 + \mathcal{O}(q - q_0)^3},$$

$$\rightarrow e^{-f(q_0)/\hbar} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2\hbar} f''(q_0) (q-q_0)^2} . \quad (\text{B.3})$$

It is straightforward to calculate the Gaussian integral in the above expression. Using the change of variable $x = (q - q_0)/\sqrt{\hbar}$, one can easily find

$$I = e^{-f(q_0)/\hbar} \sqrt{\frac{2\pi\hbar}{f''(q_0)}} e^{-\mathcal{O}(\hbar^{1/2})} . \quad (\text{B.4})$$

In the case of the action $S[x]$, which appears in the path integral calculations, one needs to use the functional derivatives. Similar to the above example, one can see that, as $\hbar \rightarrow 0$, the path integral is dominated by the classical solution

$$\frac{\delta S}{\delta x} = 0, \quad (\text{B.5})$$

with appropriate boundary condition. Therefore, for $\hbar \rightarrow 0$ only the classical solution remains.

C Wick rotation

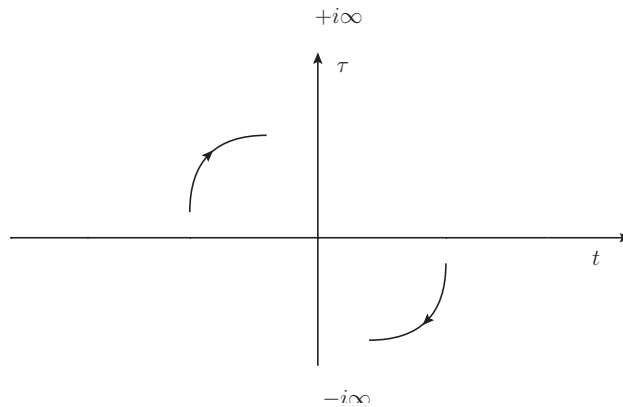
We have seen in the path integral formulation of QFT that might be useful to slightly tip into the complex plane the time integration. This corresponds to an analytic continuation from real to imaginary times. One can think of the rotation for a momentum integral (and then changing the energy of the particle into a complex energy) or for the time coordinate. In the latter case we have

$$t \rightarrow i\tau, \quad (\text{C.1})$$

that can be seen also as a rotation of the temporal axis

$$t \rightarrow e^{-i\theta} t = \tau \Rightarrow -it = \tau \Rightarrow t = i\tau, \quad (\text{C.2})$$

and for $\theta = \pi/2$ we have the Wick rotation.



D Gaussian integration: analytic continuation

We have defined functional integrals as the appropriate limit of multiple integrals. One typically encounters integrals of the type:

$$I = \int_{-\infty}^{+\infty} dx \exp(-b\epsilon x^2 + ibx^2) = 2 \int_0^{+\infty} dx \exp(-bx^2(\epsilon - i)), \quad (\text{D.1})$$

with $\epsilon \rightarrow 0$ that corresponds to the $(+i\epsilon)$ prescription.

Rotating the contour of integration $x' = xe^{i\varphi}$ so that $e^{2i\varphi}(\epsilon - i) = 1$ one gets

$$I = e^{i(\frac{\pi}{4} - \frac{1}{2}\epsilon)} \left(\frac{\pi}{b}\right)^{\frac{1}{2}} \xrightarrow{\epsilon \rightarrow 0} \left(\frac{i\pi}{b}\right)^{\frac{1}{2}}, \quad (\text{D.2})$$

which is the analytic continuation of the gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \quad (\text{D.3})$$

for complex a ($\text{Re} a > 0$).

A similar result holds for integration over the complex variable $z = x + iy$.

Gaussian integrals

From the known equation:

$$\int_{-\infty}^{+\infty} dx e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}, \quad (\text{D.4})$$

we can simply obtain

$$\int_{-\infty}^{+\infty} dx_1 \dots dx_n e^{-\frac{1}{2} \sum_{k=1}^n a_k x_k^2} = \prod_{k=1}^n \left(\frac{2\pi}{a_k}\right)^{\frac{1}{2}}. \quad (\text{D.5})$$

We can write this result using scalar products and matrices. We consider $x, y \in \mathbb{R}^n$ so that

$$x \cdot y = (x, y) = \sum_{k=1}^n x_k y_k, \quad (\text{D.6})$$

and we consider a diagonal matrix

$$A = \begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{pmatrix} \quad (\text{D.7})$$

so that

$$(x, Ax) = \sum_{k=1}^n a_k x_k^2. \quad (\text{D.8})$$

One can also define the integral measure as

$$[dx] = \frac{dx_1 \dots dx_n}{(2\pi)^{\frac{n}{2}}} = \frac{dx^n}{(2\pi)^{\frac{n}{2}}}. \quad (\text{D.9})$$

So that we can finally write

$$\int [dx] \exp \left[-\frac{1}{2}(x, Ax) \right] = \frac{1}{\sqrt{\det A}}. \quad (\text{D.10})$$

If A is not a diagonal matrix but it is symmetric, then it can always be diagonalized by an orthogonal transformation:

$$O^{-1}AO = \text{diag}(a_1 \dots a_n), \quad \text{with } O^{-1} = O^T, \quad O \in O(n). \quad (\text{D.11})$$

Then one can perform on the integral the transformation $x \rightarrow Ox$, $dx = dOx = dx$ because the determinant of O is 1 (being an orthogonal matrix). Then we write

$$\begin{aligned} \int [dx] \exp \left\{ -\frac{1}{2}(x, Ax) \right\} &= \int (dOx) \exp \left\{ -\frac{1}{2}(Ox, AOx) \right\} \\ &= \int [dx] \exp \left\{ -\frac{1}{2}(x, \text{diag}(a_1 \dots a_n)x) \right\} \\ &= \frac{1}{\sqrt{\det A}} \end{aligned} \quad (\text{D.12})$$

where the exponent has been simplified as follows

$$(Ox, AOx) = (x, O^T AOx) = (x, \text{diag}(a_1 \dots a_n)x). \quad (\text{D.13})$$

Quadratical forms

We start with the quadratic form

$$q(x) = -ax^2 + bx + c = q(x_0) - a(x - x_0)^2, \quad (\text{D.14})$$

with $x_0 = b/2a$ and hence

$$\begin{cases} \frac{\partial q}{\partial x} = -2ax + b, & \frac{\partial q}{\partial x} \Big|_{x=x_0} = 0, \\ q(x_0) = -a \frac{b^2}{4a^2} + \frac{b^2}{2a} + c = \frac{b^2}{4a} + c. \end{cases} \quad (\text{D.15})$$

Then we have

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2 + bx + c) = \exp \left(\frac{b^2}{4a} + c \right) \sqrt{\frac{\pi}{a}}, \quad (\text{D.16})$$

and we can generalize this formula to vectors $x \in \mathbb{R}^n$ and matrices $n \times n$. Then we have

$$Q(x) = \frac{1}{2}(x, Ax) + (b, x) + c \quad (\text{D.17})$$

with A positive and invertible matrix. We put

$$x_0 = -A^{-1}b \Rightarrow Q(x) = Q(x_0) + \frac{1}{2}(x - x_0, A(x - x_0)) \quad (\text{D.18})$$

and we obtain, using D.16

$$\int [dx] e^{-Q(x)} = \exp \left[\frac{1}{2}(b, A^{-1}b) + c \right] \frac{1}{\sqrt{\det A}}. \quad (\text{D.19})$$

Complex variables and Hermitian matrices

From what we have seen we can calculate

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy e^{-a \frac{(x^2+y^2)}{2}} = \frac{2\pi}{a}, \quad (\text{D.20})$$

and we can put $z = \frac{1}{\sqrt{2}}(x + iy)$, $z^* = \frac{1}{\sqrt{2}}(x - iy)$ with $z \in \mathbb{C}$, $x, y \in \mathbb{R}$. We perform then the change of variables:

$$|dxdy| = \left| \frac{\partial(x, y)}{\partial(z, z^*)} \right| dzdz^* = \left| \det \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \right| = [dz^*dz], \quad (\text{D.21})$$

so that from (D.20) we can obtain

$$\int \frac{dzdz^*}{2\pi} e^{-az^*z} = \frac{1}{a} \quad (\text{D.22})$$

and this can be generalized to

$$\int \frac{dz_1 dz_1^* \dots dz_n dz_n^*}{(2\pi)^N} e^{-(z^*, Az)} = (\det)^{-1} = e^{-\text{tr} \ln A}. \quad (\text{D.23})$$

To define gaussian integration over fields we will generalize the results that hold for N -dimensional vector spaces to infinite dimensional functional spaces. For real functions $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}$ we define the standard scalar product

$$(\phi, \psi) = \int d^4x \phi(x) \psi(x), \quad (\text{D.24})$$

and we obtain

$$\int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \int d^4x \phi(x) A \phi(x) \right\} = \frac{1}{\sqrt{\det A}}, \quad (\text{D.25})$$

$$\int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp \left\{ -\int d^4x \phi^*(x) A \phi(x) \right\} = (\det A)^{-1}, \quad (\text{D.26})$$

where the 2π factors have been included in the definition of the functional measure (this is not relevant as we will always deal with ratios of such integrals).

For any matrix that can be diagonalized by unitary transformations there are some useful formulas:

$$\begin{cases} \det(1 - A) = \exp \{ \text{tr} \ln(1 - A) \}, \\ \det A = \exp \{ \text{tr} \ln A \}, \\ \text{tr} \ln(1 - A) = -\text{tr} \left[A + \frac{1}{2} A^2 + \frac{1}{3} A^3 + \dots \right]. \end{cases} \quad (\text{D.27})$$

