Exercise 3.1: Feynman rules for Green’s functions in momentum space

In the exercise 2.1 you computed the 2-point and 4-point Green’s functions of the $\phi^4$-theory up to $\mathcal{O}(\lambda^2)$ by differentiating the generating functional with respect to the sources. In this exercise you will see that this result can be obtained in a much simpler way if you use Feynman rules for the Green’s functions in momentum space. These rules are

• Label each external point $i$ with a Latin letter, e.g. $x_i$
• Label each propagator with a different momentum
• For each propagator, $\frac{i}{p^2 - m^2 + i\epsilon}$
• For each vertex, $-i\lambda$
• For each external line, $e^{-ipx}$
• Integrate over each momentum that is not determined by the momentum conservation via $\int \frac{d^4p}{(2\pi)^4}$
• Divide by the symmetry factor $S$.

a) Draw only the connected diagrams that contribute to $G^{(2)}(x,y)$ and $G^{(4)}(x_1, x_2, x_3, x_4)$ up to $\mathcal{O}(\lambda^2)$. Green’s functions that contain only such diagrams are called connected Green’s functions.

b) For each diagram from a), use the given Feynman rules to write down the corresponding contribution to the Green’s function. Convince yourself that this gives you precisely the same contribution that you found in the exercise 2.1.
Exercise 3.2: Scattering cross section for the $\phi^4$ theory

Consider the theory of the real scalar field with quartic interaction described by the following Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (1)$$

We want to calculate the scattering of two incoming particles with the momenta $q_1$ and $q_2$ in the initial state $i$ to a final state $f$ with two outgoing particles having momenta $q_3$ and $q_4$ at the first order in $\lambda$. The Feynman rules for the $S$-Matrix elements $\mathcal{M}_{fi}$ of this theory are

- Label each propagator with a different momentum

- For each internal propagator,\n\[ p = \frac{i}{p^2 - m^2 + i\epsilon} \]

- For each vertex,\n\[ = -i\lambda \]

- For each external line,\n\[ = 1 \]

- For each loop momentum that is not determined by the momentum conservation integrate via \[ \int \frac{d^4 p}{(2\pi)^4} \]

- Divide by the symmetry factor $S$.

a) Draw all the diagrams that contribute to the 4-point Green’s function at the first order in $\lambda$ in order to obtain $\mathcal{M}_{fi}$ and $S_{fi}$. Do all of these diagrams contribute to the actual scattering process?

b) Work out the relationships between the transition amplitude $S_{fi}$, the transition probability per unit volume per unit time $W_{fi}$ and the differential cross section $d\sigma(i \rightarrow f)$ to obtain

$$d\sigma(i \rightarrow f) = \frac{\lambda^2}{64\pi^2 (q_1 + q_2)^2} d\Omega_3, \quad (5)$$

where $v_{12} = |v_1^z - v_2^z|$.

c) Perform the integration over $d^3 q_4$ and $dq_3^0$ using the relation

$$\frac{d^3 q}{2q^0} = \delta^{(4)}(q^2 - m^2)\theta(q^0) d^4 q \quad (3)$$

and the Møller covariant form

$$q_1^0 q_2^0 | v_{12} | = \sqrt{(q_1 \cdot q_2)^2 - m^4} \quad (4)$$

to obtain the result

$$d\sigma(i \rightarrow f) = \frac{\lambda^2}{64\pi^2 (q_1 + q_2)^2} d\Omega_3. \quad (5)$$
Exercise groups: time and locations

<table>
<thead>
<tr>
<th>Group</th>
<th>Tutor</th>
<th>Time</th>
<th>Room</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sungmin Hwang</td>
<td>Tuesday November 12th, 16-18</td>
<td>PH 1141</td>
</tr>
<tr>
<td>2</td>
<td>Vladyslav Shtabovenko</td>
<td>Wednesday November 13th, 8-10</td>
<td>PH 1141</td>
</tr>
</tbody>
</table>