Poincare Invariance in EFT

Sungmin Hwang

Technical University of Munich

sungmin.hwang@tum.de

May 14, 2014
Example: Expand the QCD Lagrangian in $1/M$

$$\mathcal{L}_{NRQCD} = \psi^\dagger \mathcal{O} \psi + \chi^\dagger \mathcal{O} \chi + \cdots$$  \hspace{1cm} (1)

- The operator $\mathcal{O}$ comes with Wilson coefficients.
- $\cdots$ includes terms other than bilinear ones.
- How do we fix or find the relations between the coefficients?
We can find the relations by imposing the Poincare invariance of the corresponding EFTs:

- Construct the Poincare generators of the EFTs.
- Impose the algebra conditions up to the desired order in $1/M$.
- Constraints yield relations between the coefficients.
- Calculations done up to $O(1/M)$ in NRQCD. [Brambilla, 2003]
We can go higher orders with a simpler method, using little group formalism, rather than directly expanding the Poincare generators in 1/M.

- Quantum Mechanics in Poincare group
- Quantum Field Theory in Poincare group
- QM in little group
- QFT in little group
Starting with a momentum eigenstate in Hilbert space

\[ P^\mu \psi_{p,\sigma} = p^\mu \psi_{p,\sigma}, \]  

(2)

how does it transform under Lorentz group? (i.e., \( U(\Lambda)\psi_{p,\sigma} = ? \)) From the fact

\[ P^\mu [U(\Lambda)\psi_{p,\sigma}] = \Lambda^\mu_\rho p^\rho U(\Lambda)\psi_{p,\sigma} \]  

(3)

it is natural to write

\[ U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma',\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'} \]  

(4)

Under spacetime translation: \( U(1, a)\psi_{p,\sigma} = e^{ip \cdot a}\psi_{p,\sigma} \)
Quantum Field Theory

Then how does the quantum field transform under Lorentz group?

\[ \phi_a \rightarrow M(\Lambda)_{ab} \phi_b(\Lambda^{-1}x) \] (5)

\( M(\Lambda) \) being a representation of the Lorentz group. In the infinitesimal form

\[ \delta \phi = i(a_0 h - a \cdot p - \theta \cdot j + \eta \cdot k) \phi \] (6)

and our interest lies upon the boost generator

\[ k = rh - tp \pm i\Sigma \] (7)

so that a generic quantum field transforms under the spatial boost as

\[ \phi_a(x) \rightarrow (e^{\mp \eta \cdot \Sigma})_{ab} \phi_b(B^{-1}x) \] (8)
Back to the Lorentz transformation in QM

\[ U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma',\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'} \]  

(9)

one can rewrite this in terms of the fixed reference frame, \( k \), define a standard Lorentz transformation \( L(p) \) such as \( L(p)k = p \), and define the generic momentum eigenstate in terms of the fixed reference frame

\[ \psi_{p,\sigma} \equiv U(L(p))\psi_{k,\sigma} \]  

(10)

How is the Lorentz transformation of the generic eigenstate written in terms of the reference frame? Let us do some simple computations.
Little group element (1/4)

\[
U(\Lambda)\Psi_{p,\sigma} = U(\Lambda)U(L(p))\Psi_{k,\sigma} = U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))\Psi_{k,\sigma} \equiv U(L(\Lambda p))U(W(\Lambda, p))\Psi_{k,\sigma} \quad (11)
\]

where \(W(\Lambda, p)k = k\), and it is called little group element. From the general expression of the Lorentz transformation of the states, this ”fixed state” transforms under the little group as

\[
U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma'\sigma}(W)\Psi_{k,\sigma'} \quad (12)
\]

where \(D(W)\) is a representation of the little group and hence we obtain the Lorentz transformation of the generic states in terms of the little group

\[
U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} D_{\sigma',\sigma}(W)\Psi_{\Lambda p,\sigma'} \quad (13)
\]
Little group element (2/4)

How does the little group element look like? It is necessary to answer this question in order to obtain the generators of the Lorentz group in the EFTs. Remember \( W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p) \), and we have to figure out how \( L(p) \) look like. From the fact that

\[
L(p)k = p \quad \text{for} \quad k^2 = p^2 = M^2
\]

we realize \( L(p) \) is a generalized rotation in the plane of \( k/M \equiv w \) and \( p/M \equiv v \)

\[
L(w, v)_{\mu}^{\nu} = g_{\nu}^{\mu} - \frac{1}{1 + v \cdot w} (w_{\mu} w_{\nu} + v_{\mu} v_{\nu}) + w_{\mu} v_{\nu} - v_{\mu} w_{\nu} + \frac{v \cdot w}{1 + v \cdot w} (w_{\mu} v_{\nu} + v_{\mu} w_{\nu})
\]

\[
L(w, v)_{1/2} = \frac{1 + w \cdot v}{\sqrt{2(1 + v \cdot w)}}
\]
Let us focus on the case when Lorentz transformation is an infinitesimal boost, $\Lambda = B(\eta)$ such that

$$B(\eta) v = v + \eta$$

for $1 = v^2 = (v + \eta)^2$; $v \cdot \eta = O(\eta^2)$. Thus, the boost transformation is expressed as (vector and spinor respectively)

$$B(\eta)_{\mu}{}^{\nu} = g_{\mu}{}^{\nu} - (v^\mu \eta_\nu - \eta^\mu v_\nu) + O(\eta^2)$$

$$B_{1/2}(\eta) = 1 + \frac{1}{2} \eta \gamma + O(\eta^2)$$
Therefore, the little group element for the infinitesimal boost is given by

\[
W(B(\eta), p) = L(B(\eta)p)^{-1}B(\eta)L(p) \\
= 1 + \frac{i}{2} \left[ \frac{1}{M + v \cdot p} (\eta^\alpha p_\perp^\beta - p_\perp^\alpha \eta^\beta) J_{\alpha\beta} \right] + O(\eta^2) \quad (19)
\]

where \( p_\perp^\beta \equiv p^\beta - (v \cdot p) v^\beta \) and

\[
J_{1/2}^{\alpha\beta} = \frac{i}{4} [\gamma^\alpha, \gamma^\beta] \\
(J^{\alpha\beta})_{\mu\nu} = i(g^{\alpha}_\mu g^{\beta}_\nu - g^{\beta}_\mu g^{\alpha}_\nu) \quad (20)
\]
We postulate the transformation of a quantum field theory as

\[
\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x) \tag{21}
\]

where \( W \) is a little group element as was before and in particular for the Lorentz boost,

\[
\phi_a(x) \to \exp \left[ \mp \eta \cdot \left( \frac{\Sigma \times \partial}{M + \sqrt{M^2 - \partial^2}} \right) \right]_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{22}
\]

when the reference frame is chosen as \( \nu = (1, 0, 0, 0) \).
Non-relativistic expansion (1/3)

Up until now, we have figured out the boost transformation of the relativistic quantum field by adopting the little group formalism. Let us combine this with the non-relativistic expansion so that we can apply it later to NRQCD and pNRQCD. For the $1/M$ expansion of the Lagrangian, extract the rest mass by

$$\phi_a(x) = e^{-iM_t} \phi'_a(x)$$  \hspace{1cm} (23)

and take the non-relativistic field normalization

$$\phi_a(x) = e^{-iM_t} \left( \frac{M^2}{M^2 - \partial^2} \right)^{1/4} \phi''_a(x)$$  \hspace{1cm} (24)

then how does this non-relativistic field $\phi''_a(x)$ transforms under the Lorentz boost?
Non-relativistic expansion (2/3)

From the "inverse" non-relativistic normalization

$$\phi_a''(x) = \left( \frac{M^2}{M^2 - \partial^2} \right)^{-1/4} e^{iMt} \phi_a(x) \tag{25}$$

we can deduce that

$$\phi_a''(x) \rightarrow \left( \frac{M^2}{M^2 - \partial^2} \right)^{-1/4} e^{iMt}$$

$$\times \exp \left[ +\eta \cdot \left( \frac{\Sigma \times \partial}{M + \sqrt{M^2 - \partial^2}} \right) \right]_{ab} \phi_b(B^{-1}x)$$

$$= \left( \frac{M^2}{M^2 - \partial^2} \right)^{-1/4} e^{iMt} \exp \left[ +\eta \cdot \left( \frac{\Sigma \times \partial}{M + \sqrt{M^2 - \partial^2}} \right) \right]_{ab}$$

$$\times e^{-iMt'} \left( \frac{M^2}{M^2 - \partial'^2} \right)^{1/4} \phi_b'(x') \tag{26}$$

where $x' \equiv B^{-1}x$. 
Thus, the Lorentz transformation (boost) of the non-relativistic field in $1/M$ expansion is given by

$$
\phi''(x) \rightarrow \left\{ 1 + iM \eta \cdot x - \frac{i \eta \cdot \partial}{2M} - \frac{i \eta \cdot \partial \partial^2}{4M^3} \right.
+ \frac{(\Sigma \times \eta) \cdot \partial}{2M} \left[ 1 + \frac{\partial^2}{4M^2} \right] + O(1/M^4) \right\} \phi''(B^{-1}x) \quad (27)
$$

and this is the transformation of the non-interacting and non-relativistic field. How can we implement this formalism into the interacting theory? To be continued by Matthias...
Poincare invariance constrains on NRQCD and potential NRQCD  

Johannes Heinonen, et. al (2012)  
Lorentz invariance in heavy particle effective theories  